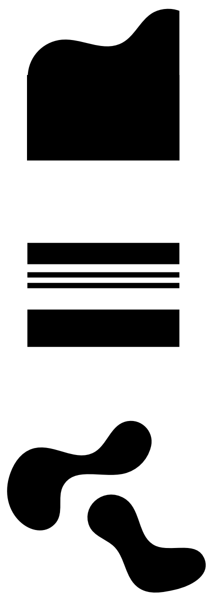
Non-measurable sets

BAS CORNELISSEN



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Bachelor's Thesis

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Abstract

Soon after Lebesgue published his theory of measures, more than one hundred years ago, sets were discovered that could not be measured. In the constructions, the Axiom of Choice played a crucial rôle. This thesis aims to investigate when and how non-measurable sets can be constructed by presenting a small selection of well-known constructions. Among them are those by Vitali, Bernstein and a partition of the plane by Sierpiński. We conclude with a set-theoretic discussion of measurable cardinals using Ulam's matrix. Each of constructions is analysed, also for the place the Axiom takes in the proofs. In this way, this thesis hopes to give an impression of what measurable sets are and what set-theoretic considerations underlie their existence.

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Contents

Preface

There are not many mathematical concepts of which one has a more lucid understanding, than one has of length, of area and of volume. Yet this entire thesis is devoted to the failure of that understanding.

Unsurprisingly, the theme has been a controversial one, which, on second thought, is surprising. Mathematics, with its rigorous, formal arguments — was it not supposed to be impervious to dispute? Perhaps not. More than one hundred years ago, a deep controversy flared up when Ernst Zermelo in 1904 published his proof of the 'Well-Ordering Theorem,' using an assumption that came to be known as Zermelo's Axiom of Choice. The Axiom was soon found to have all sorts of counter-intuitive consequences. It is no doubt one of the most striking of these, that we will examine here: non-measurable sets.

Their existence is by no means straightforward and indeed intimately connected to the Axiom of Choice. This is what I wish to look into in this text: what is needed, in set-theoretic terms, to prove their existence? And if they have come into being, what do they look like? In other words, when and how can we construct sets that have no length, no measure?

When it comes to axioms required to prove their existence, only a superficial exploration of the terrain is feasible. Similarly, I cannot present anything near a complete overview of known non-measurable sets. Instead, the discussion will center around a small selection of well-known constructions. Among them are the very first construction of such a set, by Giuseppe Vitali; a very elegant construction due to Sergei Bernstein and a partition of the plane by Wacław Sierpiński. Each of those three will be the guiding theme in one of three successive chapters. The character of the text gradually changes, allowing more and more set theory to enter the discussion, culminating in a brief, purely set-theoretic excursion to the lands of measurable cardinals, only to stop at the point where modern set theory takes over.

The text is written for students, who, like myself, have acquainted themselves only with a tiny corner of the mathematical expanse. Therefore, I have included an introductory chapter that reviews the preliminaries, hoping to make this thesis more or less self-contained. Moreover, it serves as an examination of the basic concepts underlying non-measurable sets, that will be helpful later. Many proofs are omitted in this introduction, and theorems are often only stated for later reference.

Speaking of references; there is a considerable amount of work to refer to for many of the constructions. In particular, I should mention the book by Kharazishvili (2004): a monograph on the topic containing many interesting results. It has often been helpful, but in general written at a level above my own. This is true for most books that deal with non-measurable sets or the Axiom of Choice (Oxtoby 1980, etc. Just and Weese 1995; Kanamori 2008; Jech 1973; Jech 2002 etc.). The solution was sometimes in sheer number: comparing fragments of different texts, different formulations, from different sources, shed light on difficult passages. Among these sources were also many unpublished documents, such as lecture nodes and even blogposts. These mainly served as background reading, and the final result is nearly almost based on published material. Obviously, theorems cited without proof are exclusively from published academic sources. Although a mathematical proof needs no reference, and all formulations are my own, I have sometimes included bibliographical footnotes to credit the main sources of inspiration for certain sections. Results that are directly taken from another source, are of course individually cited.

I would like to end this preface by thanking my supervisors, Benedikt Löwe and Hugo Nobrega for all their help, suggestions and valuable comments.

Notation Throughout this text I will try to use as much standard notation as possible. $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ denote the usual number systems. I will write $A \sqcup B$ for a union of disjoint sets, only for added emphasis. So I do *not* mean the 'disjoint union' $A \times \{0\} \cup B \times \{1\}$. Relative complements, whenever no confusion about the domain can arise, are abbreviated as $A^{c} := X \setminus A$ for $A \subseteq X$. Algebraic operations on sets are defined 'pointwise' so for a set A and a point x we have

$$x + A := \{x + a : a \in A\} \quad \text{and} \quad x \cdot A := \{x \cdot a : a \in A\}.$$

I write $[0, +\infty]$ for the positive part of the extended real line, i.e. $[0, +\infty] := \{x \in \mathbb{R} : x \geq 0\}$. Open intervals are sometimes abbreviated as the open ball $B_x(\varepsilon)$ with radius ε centered around a point x. The distance $d(x, A)$ between a point x and a set A is $\inf\{d(x, a) : a \in A\}$. Finally $f[A]$ and $f^{-1}[A]$ denote the image and pre-image of A under f respectively. All other notation will be introduced in due course.

CHAPTER 1

Sets and measures

This text is about non-measurability. When can we prove the existence of sets that are not measurable and how do we construct them? Clearly, two questions must be answered first: what are sets and what are measures? As to defining sets, we adopt the Zermelo-Fraenkel axioms of set theory (ZF). The real culprit is of course Zermelo's famous Axiom of Choice, which we will look into later. The entire text will assume ZF and state all additional assumptions. Concerning measures, the one of Henri Lebesgue is the canonical example and indeed the one of most interest. Although the basic theory of the Lebesgue measure is probably known, we outline its construction below. Then we consider measures from a more general point of view, trying to to identify which measures will allow for non-measurable sets in the first place.

1.1. Well-orders, ordinals and cardinals

Large parts of mathematics can be formalised in Zermelo-Fraenkel Set Theory ZF, starting with the natural numbers $\mathbb{N}$. Those can be defined as the smallest *inductive* set — the smallest set satisfying $x \in X \longrightarrow \{x\} \cup x \in X$. Using this definition, all natural numbers become sets of the following form:

$$\begin{aligned}
0 &:= \emptyset \\
1 &:= 0 \cup \{0\} = \{\emptyset\} = \{0\} \\
2 &:= 1 \cup \{1\} = \{\emptyset, \{\emptyset\}\} = \{0, 1\} \\
3 &:= 2 \cup \{2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \{0, 1, 2\} \\
&\vdots \\
n + 1 &:= n \cup \{n\} = \{0, 1, 2, 3, \dots, n - 1, n\} \\
&\vdots
\end{aligned}$$

They have two interesting properties. First is one called *transitivity*: $x \in n \in m \rightarrow x \in m$. If we then define (the usual) ordering on $\mathbb{N}$ as $n < m :\leftrightarrow n \in m$, there is a second interesting property: every nonempty subset has a least element. We will briefly show how combining these two properties leads to the notion of ordinals and cardinals, that will be required later on.

Well-orders A linear order[1] $(X, \leq)$ is called a *well-order* if it is *well-founded*, i.e. if every nonempty subset of X has a $\leq$-minimal element.[2] As mentioned, $\mathbb{N}$ is a good example of a well-order. For any partial order (X, R) (and in particular for well-orders) we define the *initial segment (determined by x)* as

$$\mathrm{IS}(x) := \{y \in X : y \leq x \text{ and } x \neq y\}$$

Whenever the ordering $\leq$ is strict, initial segments will be *proper* in the sense that they will never equal X. If there is a need to distinguish proper from non-proper initial segments, we will write $\overline{\mathrm{IS}}(x)$ for $\mathrm{IS}(x) \cup \{x\}$.

Well-orders behave very nicely. For instance, we can do *order induction*[3] on well-orders. Moreover, if a strict well-order has no greatest element, we can define a *successor function* by $s(x) := \min\{y : y > x\}$. In that way, every element has a successor, but not every element has to *be* a successor. Elements that are not successors are called *limits*. For example,

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BIBLIOGRAPHIC NOTE. This section largely follows the lectures of Prof. dr. Benedikt Löwe on axiomatic set theory.

[1] A relation is called a *linear* order if it is transitive, antisymmetric and total. To every linear order $(X, \leq)$ corresponds a *strict linear order* $(X, <)$ when we define $x < y$ iff $x \leq y \wedge x \neq y$. *Partial orders* are the 'non-total' linear orders: they are reflexive, antisymmetric and transitive.

[2] A *minimal element* m of a subset A is one such that for all b with $b \leq m$, we have $b \notin A$.

[3] In fact, precisely the well-founded relation (X, R) satisfy order induction: for all $A \subseteq X$ we have that $(\forall x(\ \mathrm{IS}(x) \subseteq A \rightarrow x \in A\ ) \longrightarrow A = X)$.

suppose that we glue two copies of $\mathbb{N}$ together in the following way:

$$0 \prec 1 \prec 2 \prec 3 \prec 4 \prec \cdots 0 \prec 1 \prec 2 \prec 3 \prec 4 \prec \cdots . \qquad (1.1)$$

Then every point is a successor, except for the second 0 (and perhaps, trivially, the first): that is a limit.

In fact, all well-orders are isomorphic to initial segments of each other. For two well-orders (X, R) and (Y, S), write $X \sqsubset Y$ if there exists an order isomorphism[4] from X to a proper initial segment of Y. Alternatively, we can write $X \cong \mathrm{IS}(y)$ for some $y \in Y$, where "$\cong$" indicates the isomorphism.

THEOREM 1.1 (FUNDAMENTAL THEOREM OF WELL-ORDERS). *For any two well-orders (X, R) and (Y, S) exactly one of the following three holds: (1) $X \sqsubset Y$; (2) $X \cong Y$; or (3) $Y \sqsubset X$.*

Proof. See Appendix A, Theorem A.3. □

Ordinals All well-orders are isomorphic to a unique element an even nicer subclass of well-orders: the ordinals. An *ordinal* is simply a transitive set well-ordered by the $\in$-relation. Our earlier observations tell us that every natural number n is an ordinal. And so is $\mathbb{N}$ itself. When thinking of $\mathbb{N}$ as an ordinal, we write $\omega := \mathbb{N}$. It is not difficult to check that the class of ordinals is closed under finite intersections and unions. Moreover, elements and initial segments of ordinals are themselves ordinals.

The ordinals are themselves almost well-ordered. For any two ordinals α and β, we always have $\alpha \subseteq \beta$ or $\beta \subseteq \beta$. Furthermore, "$\sqsubset$" coincides with "$\in$" on the class of ordinals. So exactly one of the following three is true: $\alpha \in \beta$, $\beta \in \alpha$ or $\alpha = \beta$. We normally write $\alpha < \beta$ for $\alpha \in \beta$ and $\alpha \leq \beta$ for $\alpha \subseteq \beta$. In this way the class of ordinals is effectively linearly ordered in a well-founded way.[5] What was true for well-orders, also holds for ordinals, so they come in two flavours: successors and limits. Successors are of the form $\gamma \cup \{\gamma\}$ and limits of the form $\bigcup\{\gamma : \gamma < \alpha\}$.

One of the most important properties of ordinals, is that we can do transfinite recursion (and induction) on them. Typically this consists of three steps: first we define $f(0)$, then $f(\alpha + 1)$ for successor ordinals and then $f(\lambda)$ for limit ordinals λ. It can be shown that this defines a 'function' on the class of all ordinals. Using transfinite recursion, we can then define ordinal addition, for example and write the ordinal isomorphic to (1.1) as

$$0 < 1 < 2 < 3 < 4 < \cdots \omega < \omega + 1 < \omega + 2 < \omega + 3 < \omega + 4 < \cdots$$

which we denote $\omega + \omega$.

Bijections and size Ordinals are a special class of well-orders, but we want to restrict our attention even further, to a special class of ordinals: the cardinals. First, a brief digression

[4] An *order isomorphism* is a bijection preserving the ordering.

[5] This is of course no real order since the ordinals form a proper class by the Burali-Forti theorem. Nevertheless all classes of well-orders have a least element in the following sense. If Φ is a formula describing a nonempty class of well-orders, i.e. $\exists x \varphi(x)$ and hence x is a well-order, then there is a x_0 such that $\forall z(z \sqsubset x_0 \longrightarrow \neg\Phi(z))$. In particular this is true for ordinals.

to fix our notation for the sizes of sets. We say that a set X injects into a set Y if there exists an injection $X \to Y$ and we write $X \preceq Y$. Whenever a bijection exists, we write $X \sim Y$ and we call the sets *equinumerous*. The *Cantor-Schröder-Bernstein* theorem asserts that $X \preceq Y$ and $Y \preceq X$ together imply $X \sim Y$. A set is *finite* if $X \sim n$ for some $n \in \mathbb{N}$. Whenever X is in bijection to a subset of $\mathbb{N}$ we call it *countable* and *countably infinite* if we want to stress that is is not finite. Sets that are not countable are *uncountable*. As Georg Cantor showed, there cannot exist a bijection between a set X and its power set $\mathcal{P}(X)$. In particular $\mathbb{R}$ must be uncountable, since we can show that $\mathbb{R} \sim {}^{\mathbb{N}}2 \sim \mathcal{P}(\mathbb{N})$.

Cardinals Clearly, every finite set is in bijection to some unique ordinal $n \in \omega$. Motivated perhaps by the observation that no natural number injects into a smaller natural number, we call ordinals α with the property that $\alpha \not\preceq \beta$ for all $\beta < \alpha$ *initial ordinals*. And we will give these ordinals a second name: *cardinals*. Given an ordinal α we define $\aleph(\alpha)$ to be the least ordinal that does not inject into α and we can build a sequence of aleph's ($\aleph$) from this:

$$\aleph_0 := \omega, \quad \aleph_{\alpha+1} = \aleph(\aleph_\alpha), \quad \aleph_\lambda = \bigcup\{\aleph_\gamma : \gamma < \alpha\} \quad \text{(for limits } \lambda)$$

Every $\aleph_\alpha$ is then a cardinal and vice versa. We call a cardinal $\aleph_\alpha$ a *sucessor cardinal* if α is a sucessor ordinal and a *limit cardinal* if α is a limit. When thinking of the aleph's as ordinals, we write $\omega_\alpha := \aleph_\alpha$. So ω_1, for example, is the first uncountable ordinal.

Similar as the natural numbers, the aleph's are good candidates for representing the sizes of infinite sets, so we define for any set X

$$\operatorname{Card} X := \min\{\alpha : \alpha \text{ is an ordinal and } \alpha \sim X\}.$$

The problem is that this definition only makes sense if we can find always find an ordinal α such that $\alpha \sim X$. For that, we need the Axiom of Choice, to which we will come back later.

1.2. The Lebesgue measure

Returning to non-measurable sets, we a first impression of what we mean by sets, so let us continue to the second notion we had to define formally: what is a measure?

Algebra's and additivity Let X be a set. An *algebra of sets*[6] *over* X is a collection $\mathcal{A} \subseteq \mathcal{P}(X)$ that contains X and is closed under unions and complements. A function $f : \mathcal{A} \to [0, +\infty]$ defined on an algebra of sets is called *additive* if for disjoints sets A and B we have $f(A \sqcup B) = f(A) + f(B)$. This can inductively be extended to the case of finitely many disjoint sets. If we further have $A \subseteq B$, then we can write B as a disjoint union $A \cup (B \setminus A)$ and we find that

$$f(B) = f(A) + f(B \setminus A) \geq f(A).$$

BIBLIOGRAPHIC NOTE. This section is based on Tao (2011); Capiński and Kopp (2004); Westra (2011); Oxtoby (1980)

[6] Sometimes an algebra of sets is called a *Boolean algebra* or a *ring of sets*.

In other words, every additive function is monotone with respect to inclusion. Moreover, we must have $f(\emptyset) = 0$: if not, then $\mu(\emptyset) > 0$. Since the emptyset is disjoint from itself, we reach the contradiction $\mu(\emptyset) = \mu(\emptyset \sqcup \emptyset) = \mu(\emptyset) + \mu(\emptyset)$.

If an algebra $\mathcal{S}$ is also closed under countable unions, we call it a *σ-algebra over X*. A function $g : \mathcal{S} \to [0, +\infty]$ is *countably additive* if for any collection $\{A_i : i \in \mathbb{N}\}$ of pairwise disjoint sets in $\mathcal{S}$ we have

$$g\Big(\bigsqcup_{i\in\mathbb{N}} A_i\Big) = \sum_{i\in\mathbb{N}} g(A_i).$$

By the same argument as before we also have $g(\emptyset) = 0$. Now let A_1 and A_2 be disjoint sets and set $A_n = \emptyset$ for all $n \geq 3$ to see that any countably additive function is also additive.

It is easily seen that the intersection of a collection of σ-algebra's is again a σ-algebra. It thus makes sense to define, for any collection $\mathcal{F} \subseteq \mathcal{P}(X)$, the *$\sigma$-algebra $\sigma(\mathcal{F})$ generated by $\mathcal{F}$* as the intersection of all σ-algebra's containing $\mathcal{F}$. An important example is the *Borel σ-algebra $\mathcal{B}$* generated by all open intervals of $\mathbb{R}$ (or more generally, the open sets of any topological space).

Measures Now we can give the definition of a measure.

DEFINITION 1.2. Let $\mathcal{S}$ be a σ-algebra over a set X. A *measure* is a function $\mu : \mathcal{S} \to [0, +\infty]$ that is[7] not identically zero on $\mathcal{S}$ and countably additive, i.e. for pairwise disjoint sets $\{A_i : i \in \mathbb{N}\}$ we have that $\mu\big(\bigsqcup_{i\in\mathbb{N}} A_i\big) = \sum_{i\in\mathbb{N}} \mu(A_i)$.

Note that non-zeroness is equivalent to $\mu(X_{>}0)$ by monotonicity. Many other conditions can be attached, to obtain different kinds of measures and we discuss some of them. If we weaken countable additivity and demand μ to be *(finitely) additive* instead, that is how we will denominate the measure as well. Countable additivity can also be strengthened to so called *κ-additivity* and we will come back to this point later. We call a measure *infinite* if $\mu(X) = \infty$ and *finite* if $\mu(X) < \infty$. If μ is a finite measure on X, it can always be normalized to

$$\hat{\mu}(A) := \frac{\mu(A)}{\mu(X)}, \qquad A \subseteq X,$$

such that $\mu(X) = 1$. Measures with the property $\mu(X) = 1$ are called *probability measures*. We have just shown that every set with a finite measure allows a probability measure. Sometimes a space with infinite measure can be approximated by sets of finite measure. In this case, when there is a countable collection $\{A_n : n \in \mathbb{N}\}$ of increasing sets with finite measure, such that $\bigcup_{n\in\mathbb{N}} A_n = X$, then the measure on X is called *σ-finite*.

Usually, the triple (X, Σ, μ) is called a *measure space* and the pair (X, Σ) a *measurable space*. Since $\mathcal{P}(X)$ is a σ-algebra, we could hope to find a measure that is defined on all subsets of a set X, i.e. on $\mathcal{P}(X)$. Such a measure is called *total*.

[7] Often the condition $\mu(\emptyset) = 0$ is included, but indeed this is true for any (countably) additive function.

From outer measures to measures Only a rough outline of the construction of the Lebesgue measure is given, since most theory is probably known. The starting point for measuring subsets of $\mathbb{R}$ is the *Lebesgue outer measure* on $\mathbb{R}$, defined by

$$m^*(A) := \inf\Big\{\sum_{n\geq 1} \ell(I_n) : \{I_n\} \text{ are intervals covering cover } A\Big\}, \quad A \subseteq \mathbb{R}$$

where $\ell([a,b]) := b - a$ denotes the length of an interval. The outer measure can be shown to have several nice properties[8]. Intervals are assigned their length and $m^*(\emptyset) = 0$. Such sets, with outer measure zero, are called *null sets*. Furthermore m^* is monotone with respect to inclusion and countably *sub*additive. That is: if $\{A_n : n \in \mathbb{N}\}$ is a collection of (not necessarily disjoint) subsets of X, then

$$\lambda(\bigcup_{n\in\mathbb{N}} A_n) \leq \sum_{n\in\mathbb{N}} \lambda(A_n)$$

To turn the countable subadditivity into countable additivity, one approach is to restrict the domain of definition of the outer measure. Therefore consider the sets $E \subseteq X$ that satisfy *Carathéodory's Condition*:

$$\lambda(A) = \lambda(A \cap E) + \lambda(A \cap E^c), \quad \text{for all } A \subseteq X. \tag{1.2}$$

The following theorem says that this restriction works:

THEOREM 1.3 (CARATHÉODORY EXTENSION THEOREM). *Let λ be an outer measure on X and Σ_λ the collection of sets satisfying Carathéodory's Condition* (1.2). *Then Σ_λ is a σ-algebra and $\mu := \lambda \restriction \Sigma_\lambda$ is a measure.*

Proof. See for example Tao (2011), Theorem 1.7.3. □

Applying Theorem 1.3 to the Lebesgue outer measure m^* yields the *Lebesgue measure*, which we will exclusively denote by m. Sets that satisfy (1.2) with respect to m^* will be called *Lebesgue measurable* and the σ-algebra of these sets is denoted by $\mathcal{M}$, so we have constructed the measure space $(\mathbb{R}, \mathcal{M}, m)$. Whenever we speak of measurability without further reference to any specific measure, we always refer to the Lebesgue measure.

From open sets to measures Although this presentation of the Lebesgue measure might be relatively short, it does not appeal to geometric intuition. For example, *Littlewoods First Principle*[9] states that measurable sets are nearly open sets and there is a characterisation of measurable sets following this intuition:

THEOREM 1.4. *A set $E \subseteq \mathbb{R}$ is Lebesgue measurable if and only if for every $\varepsilon > 0$ there is a closed set F and an open set G such that $F \subseteq A \subseteq G$ and $m^*(G \setminus F) < \varepsilon$.*

Proof. See for example Capiński and Kopp 2004, p.43-44. □

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[8] e.g. Oxtoby 1980, Theorem 3.1, 3.2 and 3.3.
[9] Tao 2011, Remark 1.2.3.

The fact that measurable sets are nearly open sets, suggests a connection between the measurable sets $\mathcal{M}$ and the Borel σ-algebra $\mathcal{B}$, the smallest σ-algebra containing the open intervals $\mathcal{I}$. Since open intervals are measurable, it follows immediately that $\mathcal{B} \subset \mathcal{M}$. It is easily to verify that subsets of null sets satisfy the Carathéodory condition and are thus in $\mathcal{M}$. Whenever a σ-algebra is closed under subsets of null sets, we call the measure space *complete*. $\mathcal{M}$ is complete, but $\mathcal{B}$ is not. In fact,[10] $\mathcal{M}$ is the completion of $\mathcal{B}$: the smallest complete σ-algebra containing $\mathcal{B}$.

Measures μ defined on $\mathcal{B}$ are called *Borel measures* and when they satisfy

$$\mu(B) = \inf\{\mu(O) : O \supseteq B \text{ open}\} \quad \text{or} \quad \mu(B) = \sup\{\mu(F) : F \subseteq B \text{ closed}\}$$

we call them *outer* or *inner regular* respectively. A *regular* measure satisfies both and the Lebesgue measure is an example of a regular measure.

To see this, first note that we can[11] characterise the outer measure in topological terms:

$$m^*(A) = \inf\{m(G) : G \text{ is an open set containing } A\},$$

which naturally suggests the converse notion of an *inner measure* $m_*(A)$:

$$m_*(A) := \sup\{m^*(F) : F \text{ is a bounded closed set containing } A\}$$

Intuition moreover suggests that sets for which these coincide, should be measurable (which happens to be very close to the original definition of measurable sets adopted by Lebesgue[12]).

THEOREM 1.5. *If A is a (Lebesgue) measurable set then $m_*(A) = m^*(A) = m(A)$. Conversely, if A is bounded and $m_*(A) = m^*(A)$, then A is measurable.*

Proof. See Oxtoby (1980), Theorem 3.18. □

We finish the section with listing some properties of the Lebesgue measure that will be used throughout the text.

PROPOSITION 1.6. *The Lebesgue measure on $\mathbb{R}$ satisfies the following properties.*

(i) (Monotonicity) If $A \subseteq B$, then $m(A) \leq m(B)$.

(ii) (Intervals) $m(I) = \ell(I)$ for every interval I.

(iii) (Complements) Whenever $A \subseteq B$ are measurable sets and $m(A) < \infty$, we have $m(B \setminus A) = m(B) - m(A)$.

(iv) (Translation invariant) $m(A + x) = m(A)$ for all $A \in \mathcal{M}$ and $x \in \mathbb{R}$.

[10] Capiński and Kopp 2004, Theorem 2.28.
[11] e.g. Oxtoby 1980, Theorem 3.18.
[12] Lebesgue 1902.

1.3. The Axiom of Choice

Having seen the basic theory of the Lebesgue measure, we return to sets. The theory of cardinality is a strange one, when developed on the basis of ZF only and for it to work nicely, we need the Axiom of Choice. For present purposes, the Axiom is also interesting because of its crucial rôle in the construction of non-measurable sets. Let us first state it.

Axiom of Choice (AC)**.** For every set X of nonempty sets there exists a *choice function*, i.e. a function $c : X \to \bigcup X$ such that $c(x) \in x$ for all $x \in X$.

There are many sorts of 'choice': statements that are weaker — that is, implied by — or stronger than the Axiom of Choice in ZF. As we would later like to find a 'minimum' amount of choice[13] needed to construct certain non-measurable sets, we analyse the following versions of choice and some related statements:

WO	(Well Ordering Principle.) Every set can be well-ordered.
CC	(Cardinal Comparison.) For all x and y we have $x \preceq y$ or $y \preceq x$.
ZL	(Zorn's Lemma.) Every partial ordered set $(P, \leq)$ with the property that every chain in P has an upper bound in P, has a maximal element.[14]
VSB	Every vector space has a basis.[15]
DC	(Axiom of Dependent Choice.) Let R be a relation on a nonempty set X. If for every $x \in X$ there exists an x' with xRx', then there exists a sequence $\{x_n \in X : n \in \mathbb{N}\}$ such that $x_n R x_{n+1}$ for all $n \in \mathbb{N}$.
AC_I	Every collection $\{X_i : i \in I\}$ of nonempty sets indexed by I has a choice function.
UC_I	The union of a set $\{X_i : i \in I\}$ of countable sets X_i injects into I.
$\mathsf{CUC}(X)$	X is the countable union of countable sets[16]
$\mathsf{VSB}_{\mathbb{Q}}(\mathbb{R})$	There exists a (Hamel-)basis for $\mathbb{R}$ as $\mathbb{Q}$-vectorspace.
$\omega_1 \preceq \mathbb{R}$	There exists an injection $\omega_1 \to \mathbb{R}$.

In general, when we append "(X)" to the name of an axiom, we restrict the applicability of the axiom to subsets of X. For example: $\mathsf{AC}(X)$ says that there exists a choice function for every set of nonempty *subsets of* X.

It is well-known[17] that the first versions of choice are equivalent:

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[13] Here 'minimum' should not be read in a strict sense: I will only present the weakest form of the ones presented here allows the construction.

[14] A *chain* is a totally ordered subset of P and an *upper bound* of some subset $S \subseteq P$ is an element $b \in P$ such that $(\forall s \in S)(s \leq b)$. Finally, a *maximal element* is an element $m \in P$ such that $p \leq m$ for all $p \in P$.

[15] Briefly, a basis for a vector space is a set that spans the space and has the property that every *finite* subset is linearly independent. We will look at this in more detail in section 2.3.

[16] Sometimes, the abbreviation $\mathsf{CUC}(X)$ is used for "Countable unions of countable subsets of X are countable". That would be $\mathsf{UC}_\omega(X)$ in our notation.

[17] e.g. Enderton 1977, Theorem 6M; Devlin 2000, Theorem 2.7.3.

THEOREM 1.7. $\mathsf{ZF} \vdash (\mathsf{AC} \longleftrightarrow \mathsf{WO} \longleftrightarrow \mathsf{CC} \longleftrightarrow \mathsf{ZL})$

In fact, one equivalence could be added to this list. It was shown by Andreas Blass[18] using an ingenious proof that $\mathsf{ZF} \vdash (\mathsf{VSB} \longrightarrow \mathsf{AC})$. The converse is much easier to establish.

THEOREM 1.8. $\mathsf{ZF} \vdash (\mathsf{AC} \longrightarrow \mathsf{VSB})$.

Proof. The proof uses Zorn's Lemma ZL. Let $\mathcal{E}$ denote the linearly independent subsets of V, ordered by inclusion and let $\mathcal{C}$ be a chain in $\mathcal{E}$. Then $C := \cup\mathcal{C}$ is an upper bound and it must be independent. Otherwise it would contain a linearly *dependent* set $e_1, \dots e_n$, each of which is contained in some $C_i \in \mathcal{C}$. Since $\mathcal{C}$ is totally ordered, there exists a $C_i \in \mathcal{C}$ that contains all of them, contradicting the independence of C_i. This shows that every chain in $\mathcal{E}$ has an upper bound in $\mathcal{E}$ and we may apply Zorn's Lemma to find a maximal element $E \in \mathcal{E}$. Indeed, E is the basis we are looking for. It is linearly independent, since it is an element of $\mathcal{E}$, so it suffices to show it spans V. Suppose otherwise and find a vector x outside Span E. Then $E \cup \{x\}$ is a linear independent subset of V, contradicting E's maximality. □

For the specific case of $\mathbb{R}$ as a $\mathbb{Q}$-vectorspace, we can adapt this proof to see that $\mathsf{ZF} \vdash \mathsf{WO}(\mathbb{R}) \longrightarrow \mathsf{VSB}_{\mathbb{Q}}(\mathbb{R})$. We later return to the proof, in Theorem 2.12. First we investigate the other statements.

THEOREM 1.9. *If* $X \preceq Y$ *then* $\mathsf{ZF} \vdash (\mathsf{AC}_Y \longrightarrow \mathsf{AC}_X)$.

Proof. Let $\{A_x : x \in X\}$ be a set of nonempty sets and $f : X \to Y$ an injection. For every $y \in Y$, define $B_y := f[A_x]$ if $f(x) = y$ and $B_y := \{y\}$ otherwise. Then we can find a choice function c_Y on $\{B_y : y \in Y\}$ by $\mathsf{AC}(Y)$. Note that c_Y is indeed defined on each $f[A_x]$ so we define

$$c_X(A_x) := (f \restriction A_x)^{-1}\Big[c_Y\big(f[A_x]\big)\Big].$$

This is a choice function for $\{A_x : x \in X\}$. □

THEOREM 1.10. *Let* κ *be an infinite cardinal, Then* $\mathsf{ZF} \vdash (\mathsf{AC}_\kappa \longrightarrow \mathsf{UC}_\kappa)$.

Proof. Let $\{A_\gamma : \gamma < \kappa\}$ be a set of countable sets A_γ and let $A = \bigcup_{\gamma<\kappa} A_\gamma$. For each A_γ there exists a set F_γ of injections $A_\gamma \to \omega \subseteq \kappa$. Use AC_κ to select one f_γ from each of the κ-many sets F_γ. Then define the function

$$\Phi : \quad A \longrightarrow \kappa \times \kappa : \quad \gamma \longmapsto (\nu(\gamma),\, f_{\nu(\gamma)}(\gamma)),$$

where $\nu(\gamma)$ denotes the smallest $\alpha < \gamma$ such that $\gamma \in A_\alpha$. Two different points α and β are mapped to the same point if and only if $\nu(\alpha) = \nu(\beta) =: \nu$ and $f_\nu(\alpha) = f_\nu(\beta)$. But this is impossible, since f_ν is an injection. So Φ is an injection from A to $\kappa \times \kappa$, which is equinumerous to κ and the proof is complete. □

An important case of this theorem is $\kappa = \omega$, for which UC_ω asserts that countable unions of countable sets are countable (the *Countable Union Theorem*).

[18] Blass 1984.

THEOREM 1.11. $\mathsf{ZF} \vdash (\mathsf{AC} \longrightarrow \mathsf{DC} \longrightarrow \mathsf{AC}_\omega)$.

Proof. If R is a relation on X such that for every x there is a x' with xRx', define a set $A_x := \{y \in X : xRy\}$ for every $x \in X$. By AC there exists a choice function c for the set $\{A_x : x \in X\}$ with $c(x) \in A_x$ for all x. Use that to define a a function $f(x) := c(A_x)$ and then define the sequence

$$x,\ f(x),\ f(f(x)),\ f(f(f(x))), \ldots,\ f^n(x), \ldots.$$

which proves DC.

Let $\{A_n : n \in \mathbb{N}\}$ be a countable collection of sets and let $\mathcal{F}$ be the set of all finite sequences $(a_0, a_1, \ldots, a_n)$ such that $a_i \in A_i$ for all $0 \leq i \leq n$. We define a relation R on $\mathcal{F}$ by aRa' if and only if $a = (a_0, \ldots, a_n)$ and $a' = (a_0, \ldots, a_n, a_{n+1})$. This satisfies the conditions of DC so there exists a sequence that selects exactly one element from each A_n. □

THEOREM 1.12. *The following statements hold.*

(i) $\mathsf{ZF} \vdash (\mathsf{AC} \longrightarrow \mathsf{AC}(X) \longrightarrow \mathsf{AC}_I(X))$.

(ii) $\mathsf{ZF} \vdash (\mathsf{AC} \longrightarrow \mathsf{AC}_I \longrightarrow \mathsf{AC}_I(X))$.

(iii) $\mathsf{ZF} \vdash (\mathsf{UC}_\kappa \longrightarrow \mathsf{UC}_\kappa(X))$

(iv) $\mathsf{ZF} \vdash (\mathsf{AC}(X) \longleftrightarrow \mathsf{WO}(X))$.

(v) $\mathsf{ZF} \vdash (\mathsf{UC}_\omega(\mathbb{R}) \longrightarrow \neg\mathsf{CUC}(\mathbb{R}))$

(vi) $\mathsf{ZF} \vdash (\mathsf{WO}(\mathbb{R}) \longrightarrow (\omega_1 \preceq \mathbb{R}))$.

Proof. The first three statements are trivially true. The proof of (iv) is very similar to the proof of $\mathsf{AC} \longleftrightarrow \mathsf{WO}$; see Appendix A Theorem A.4 Then (v) is almost immediate: if $\mathsf{CUC}(\mathbb{R})$ holds, then $\mathbb{R}$ is countable by $\mathsf{UC}_\omega(\mathbb{R})$. For (vi), well-order $\mathbb{R}$ and note that by the Fundamental Theorem of Well-orders (Theorem 1.1) $\omega_1 \sqsubset \mathbb{R}$, since ω_1 is by definition the smallest uncountable ordinal. More precisely, if $\mathbb{R} \sqsubset \omega_1$ then there exists $\gamma \in \omega_1$ such that $\mathbb{R} \cong \mathrm{IS}(\gamma)$. $\mathrm{IS}(\gamma)$ must be infinite and by definition of ω_1, it injects into ω. It follows that $\mathrm{IS}(\gamma)$, hence $\mathbb{R}$, is countable, which is not true. □

Some special cases of these relations are illustrated in Figure 1.1.

1.4. Conditions for non-measurability

Lebesgue developed his measure theory in his 1902 thesis. He started with proposing the following *Measure Problem*: is it possible to define a nonzero, translation invariant and countably additive measure on $\mathbb{R}$? He succeeded, as we have seen, to do this for the Lebesgue measurable sets $\mathcal{M}$, but the question remains if it is possible to measure *all* subsets of $\mathbb{R}$. In other words, if there exists a *total* translation invariant measure on $\mathbb{R}$. Since this text is about *non*-measurability, the answer is probably negative. Here want to investigate what properties a measure should have to leave room for non-measurable sets and

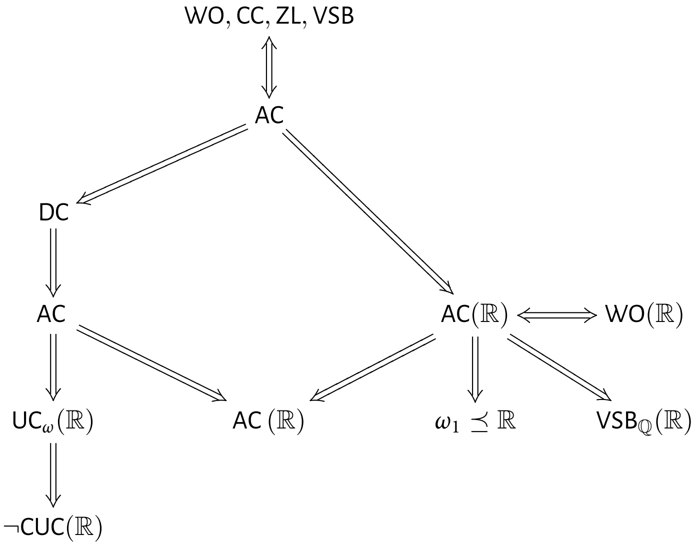

Figure 1.1.: Relations between different versions of AC. All double arrows indicate a (not necessarily strict) implication, provable in ZF.

what properties $\mathbb{R}$ should have. To put it differently: when can we *not* prove the existence of non-measurable sets?

We start by consider three pathological measures.

Example 1.13 (Counting Measure)**.** On a nonempty set X we define the *counting measure* by

$$\#A = \begin{cases} n & \text{if } A \sim n \text{ for some } n \in \mathbb{N} \\ \infty & \text{otherwise,} \end{cases} \qquad A \subseteq X$$

This is easily seen to be a countably additive measure. On $X = \mathbb{R}$ it is also translation invariant and defined for every subset. Consequently it is a solution to the measure problem.

Example 1.14 (Dirac measure)**.** Let X be a set and $a \in X$ some a fixed element. Then define the *Dirac measure*

$$\delta_a(A) = \begin{cases} 1 & \text{if } a \in A \\ 0 & \text{otherwise} \end{cases}, \qquad A \subseteq X.$$

Clearly, it is nonzero, but also countably additive: at most one set in a collection of *disjoint* sets can contain a and if so, the sum of the measures will thus equal 1. The Dirac measures is also a total measure, but it is not a solution to the measure problem since it is not translation invariant.

Example 1.15 (The 0-∞-measure)**.** Let X be an infinite set and define the 0-∞-*measure* $\mu_{0\infty}$ to be 0 for all finite sets, and ∞ otherwise. This is strictly speaking not a measure, since countable additivity clearly fails. *Finite* additivity does hold, on the other hand.

We want to exclude these three measures for two reasons. Firstly, they are all total measures. Without further conditions, general measures will thus be completely uninteresting with regard to non-measurability. Secondly, these three measures, on $\mathbb{R}$, have no connection to the length of intervals and certainly do not extend the Lebesgue measure. For example, both the counting measure and the 0-∞-measure assign every proper interval infinite measure.

Lebesgue eventually changed his formulation of the Measure Problem[19] and replaced the condition that μ is not identically zero by the condition $\mu([0, 1]) = 1$. More generally, we could only consider probability measures, i.e. satisfying $\mu(X) = 1$. This would already exclude the counting measure and the 0-∞-measure, but still allows the Dirac measure on $X = [0, 1]$.

Another property that the first two measures lack, is *difusseness*. Measures that satisfy $\mu(\{x\}) = 0$ for all x are called *diffuse measures*[20]; all others *non-diffuse*. Equivalently, one could say of a diffuse measure that it *vanishes at singletons*, for obvious reasons. We have seen that the counting measure and the Dirac measure are non-diffuse. But only requiring measures to be diffuse, does not exclude the 0-∞-measure. Admittedly, this is not a real measure — it is finitely additive — but that is no surprise:

PROPOSITION 1.16. *There exist no non-diffuse finite measures on infinite sets.*

Proof. Let X be an infinite set and μ a non-diffuse finite measure on X. Write $m := \mu(\{x\}) > 0$, find a $k \in \mathbb{N}$ such that $k \cdot m > \mu(X)$ and let A be a subset of X containing at least k points. Then we have $\mu(A) = k \cdot m > \mu(X)$ which contradicts monotonicity of μ. □

The three pathological measures can thus be excluded by ignoring non-diffuse infinite measures. In other words: the interesting measures are diffuse probability measures, which we will give a special name[21].

DEFINITION 1.17. A *weak measure* is a diffuse probability measure.

Indeed, the Lebesgue measure on $[0, 1]$ is a weak measure, but we could wonder if we did not move too far from the original Measure Problem by restricting ourselves to finite measures. This theorem gives an answer.

THEOREM 1.18. *There exists a total, translation invariant measure on* $\mathbb{R}$ *if and only if there exists a measure defined on* $\mathcal{P}([0, 1)]$ *that is translation invariant on* $[0, 1)$*: if* A *and* $A + x$ *are subsets of* $[0, 1)$*, then their measures coincide.*

---

[19] Moore 1983, p. 137.

[20] The terminology is from Kharazishvili (2004). Diffuse measures are related to *atomless measures*. An *atom* is a set of positive measure such that every subset with strictly smaller measure, has measure zero. An atomless measure is one without atoms. For diffuse measures, singletons are never atoms.

[21] cf. Pawliuk 2010.

Proof. One direction is immediate: a measure on $\mathbb{R}$ can be restricted to $[0, 1]$. Conversely, suppose that $\mu : \mathcal{P}([0, 1)) \to [0, +\infty]$ satisfies the conditions. Define the measure of a set $A \subseteq \mathbb{R}$ by measuring its parts in each interval $[n, n + 1)$:

$$\mu_{\mathbb{R}}(A) := \sum_{n\in\mathbb{N}} \mu\Big(([z_n, z_n + 1) \cap A) - z_n\Big),$$

where $z_0, z_1, \ldots$ is an enumeration of $\mathbb{Z}$. The proof that this works is not difficult and consists mainly of technicalities that are of little value to our current discussion. Therefore it can be found in Appendix A Theorem A.5. □

This justifies the *Generalized Measure Problem*, first introduced by Banach[22] which roughly asks for a weak measure defined on all subsets of a set X. Rephrasing this slightly, which sets X allow a weak measure? This question leads directly to deep set-theoretic considerations, which we will not touch upon until chapter 5.

For now, we stay closer to the lands of the concrete and try to further fence off the realm of non-measurability. Clearly, the possibility of non-measurable sets als depends on the the structure of $\mathbb{R}$. More precisely, it depends on the truth of:

$\mathsf{CUC}(\mathbb{R})$. $\mathbb{R}$ is the countable union of countable sets.

THEOREM 1.19. *Suppose* $\mathsf{CUC}(\mathbb{R})$ *holds. Then there is no countably additive measure on* $\mathbb{R}$.

Proof. Suppose otherwise; let μ be a countably additive measure. By $\mathsf{CU}(\mathbb{R})$ we can write $\mathbb{R} = \bigcup\{R_i : i \in \mathbb{N}\}$ where all R_i are countable. They can moreover be assumed to be disjoint (by defining $\hat{R}_{i+1} := R_{i+1} \setminus \hat{R}_i$). Since μ is indiscrete, every countable set, and in particular ever R_i, is μ-measurable and has measure zero. It follows that $\mu(\mathbb{R}) = \sum_{i\in\mathbb{N}} \mu(R_i) = 0$ and by monotonicity μ must be identically zero, which is impossible. □

So is $\mathsf{CUC}(\mathbb{R})$ true? Well, as seen in Theorem 1.12 (v), not if the $\mathsf{UC}_\omega(\mathbb{R})$ is true. Nevertheless it has been shown by Feferman and Levy to be consistent with ZF:

THEOREM 1.20 (10.6 IN JECH 1973). *There exists a model of* ZF *where* $\mathsf{CUC}(\mathbb{R})$ *holds.*

The details of this theorem are of course beyond the scope of this thesis. It is only reproduced here, because together with Theorem 1.19, it shows that ZF alone does not even proof the countable additivity of the Lebesgue measure. So a satisfactory definition requires some amount of choice. On the other hand, there is a sort of 'lower bound' on the amount of choice needed to prove the existence of non-measurable sets:

THEOREM 1.21 (10.10 IN JECH 1973). *There exists a model of* ZF *which satisfies the Axiom of Dependent Choices (*DC*) and in which every set of reals is Lebesgue-measurable.*

If there is a model of ZF + DC in which every set is measurable, then ZF + DC surely does not prove the existence of non-measurable sets. To do that, we will have to use stronger

[22] Moore 1983, p. 147.

versions of the axiom of choice — at least if we are looking for countably additive measures. Indeed, weakening that condition makes room for a total *additive* measure extending Lebesgue's, as was shown by Banach. His proof uses the well-known Hahn-Banach theorem which roughly says that bounded linear functionals defined on a subset of a larger normed space can, under certain conditions, be extended to the whole space.

THEOREM 1.22 (BANACH). *If the Hahn-Banach theorem is true, there exists a finitely additive measure defined for all subsets of $\mathbb{R}$ and extending the Lebesgue measure.*

We will not prove this theorem here[23]. It involves the lengthy construction of the so called *Banach Integral* that can be used to define a measure. The result is relevant to our current discussion beacuse it shows that ZF with the Hahn Banach theorem[24] does not prove the existence of non-measurable sets with respect to *finitely additive* measures.

Conclusion This chapter set out to discuss the concepts that underly non-measurable sets: sets and measures. Having introduced sets, the Lebesgue measure and the Axiom of Choice, we investigated three pathological measures. Each of them was a total measure of $\mathbb{R}$, which means that without further conditions ZF does not prove the existence of non-measurable sets, even if we add more choice. Therefore we only have to look at diffuse measures that $\mu([0,1]) = 1$ on $\mathbb{R}$, or more generally $\mu(X) = 1$. Finitely additive measures also have no hope to bring us non-measurable sets, since in ZF with the Hahn-Banach theorem, there exists a finitely additive extension of the Lebesgue measure. Moreover, in ZF alone we cannot prove the existence of countably additive measures. Taking this all together, we conclude that as it comes to proving the existence of non-measurable sets, we only have to consider countably additive, weak measures in the axiomatic system of ZF with some version of choice added. Moreover, this must be more than DC as a model of ZF + DC exists where every set is Lebesgue-measurable.

It is time to move on and start building non-measurable sets.

[23] See for example Bachman and Narici 2012, Appendix 2.

[24] It should be noted that the Hahn-Banach theorem is itself a weak form of choice (e.g. Jech 1973, p.28). In fact, there exists a model of ZF where it fails to be true (The Feferman/Pincus model, see Howard and Rubin 1998, p. 149). It has furthermore been shown that ZF plus the Hahn-Banach theorem implies the existence of a non-measurable set (Foreman and Wehrung 1991).

CHAPTER 2

Measuring reals and rationals

The first and best known construction of a set that is not Lebesgue measurable was given in 1905 by the Italian mathematician Giuseppe Vitali.[1] He showed that it was impossible to define a translation invariant measure on all subsets of $\mathbb{R}$, thereby answering Lebesgue's measure problem. We will present the construction and then analyse it in more detail by first generalising it towards the notion of a set of Vitali-type. Then we will look at their relation to dense subsets. The last section shows that a Vitali-type set can be constructed from a Hamel bases of $\mathbb{R}$ as $\mathbb{Q}$-vectorspace.

BIBLIOGRAPHIC NOTE. Generalizations of the Vitali construction are inspired by Kharazishvili (2004)

[1] Vitali 1905.

2.1. Vitali's non-measurable sets

THEOREM 2.1 (VITALI, 1905). *Assume* $\mathsf{WO}(\mathbb{R})$ *and let* μ *be a translation invariant measure on* $\mathbb{R}$ *satisfying* $\mu([0,1)) = 1$. *Then there exists a subset of* $\mathbb{R}$ *that is not measurable with respect to* μ. *In particular, there exists a non-Lebesgue measurable subset of* $\mathbb{R}$.

Proof. Consider $\mathbb{R}$ with the equivalence relation $x \sim y$ if and only if $x - y \in \mathbb{Q}$. By $\mathsf{WO}(\mathbb{R})$, we can find a selector V for this equivalence relation, that is moreover a subset of $[0,1)$: pick for every x the least element in $[0,1)$ equivalent to x. This can always be done, since every point is equivalent to its decimal part. Note that the equivalence classes correspond to the elements of the quotient group $\mathbb{R}/\mathbb{Q}$; they are of the form $x + \mathbb{Q}$. From this we see that the difference $v - w$ of two unequal elements in V must be irrational.

Now assume towards a contradiction that V is measurable. Any $x \in \mathbb{R}$ is equivalent to some $v \in V$ and can thus be written as $x = q + v$ for some rational q. In other words, the collection

$$\{q + V : q \in \mathbb{Q}\}.$$

of rational translates of V covers $\mathbb{R}$. All sets in this collection have the same measure $\mu(V)$ since μ is invariant under translation. Moreover, any two sets $q + V$ and $p + V$ are disjoint. After all, a point x in their intersection would be of the form $x = q + v = p + w$ with p and q rationals and $v, w \in V$. It follows that $v - w = p - q$ is rational. We observed earlier that unequal points in V have an irrational difference, so we conclude that $v = w$ and $p + V = q + V$. Countable additivity and translation invariance of the measure give

$$1 \leq \mu(\mathbb{R}) = \mu\Big(\bigsqcup_{q\in\mathbb{Q}} q + V\Big) = \sum_{q\in\mathbb{Q}} \mu(q+V) = \sum_{q\in\mathbb{Q}} \mu(V),$$

so we must have $\mu(V) > 0$. To summarize, we have found a countable collection of disjoint translates of V each of identical nonzero measure.

Now consider the infinite union over $\mathbb{Q} \cap [0,1)$:

$$X = \bigsqcup \big\{q + V : q \in \mathbb{Q} \cap [0,1)\big\}.$$

Since $V \subseteq [0,1)$, this must be a subset of $[0,2)$. Monotonicity of μ implies that $\mu(X) \leq \mu([0,2)) = \mu([0,1)) + \mu([0,1) + 1) = 2$. But at the same time

$$\mu(X) = \sum_{q\in\mathbb{Q}\cap[0,1)} \mu(q+V) = \infty$$

since all $q + V$ have positive measure. This contradiction shows that V cannot be μ-measurable after all. ☐

We can also read the proof differently. Suppose that μ is an extension of the Lebesgue measure to $\mathcal{P}(\mathbb{R})$ and further suppose that μ is translation invariant. Then the set V must be μ-measurable, and we reach the same contradiction. In other words:

PROPOSITION 2.2. *If there exists an extension of the Lebesgue measure to* $\mathcal{P}(\mathbb{R})$, *it cannot be translation invariant.*

So much for the measure; let us consider the sets in more detail. It is easily seen that Vitali's construction can be generalised.

THEOREM 2.3. *Let (X, Σ, μ) be a measure space where X is equipped with a commutative group structure and let G be a countably infinite subgroup of X. Suppose V is a selector for X/G such that the following conditions are met.*

(i) The measure μ is invariant under G. That is, for all $g \in G$ and μ-measurable sets A, the translate $g + A$ is again measurable and $\mu(A) = \mu(g + A)$.

(ii) There exists a set $C \subseteq X$ that contains V and has an infinite intersection with G, such that $C + C$ is measurable and has finite measure.

Then V is not μ-measurable.

Proof. The proof closely follows Vitali's construction, so assume V is measurable. By the same arguments as before, the sets $\{g + V : g \in G\}$ form a disjoint cover of X. All these sets have the same measure by G-invariance. Because they cover X, it must be positive, so $\mu(V) > 0$. It follows immediately that $\mu\big(\bigsqcup\{g + V : g \in G \cap C\}\big) = \infty$. However, this union must also be 'bounded' in the sense that

$$\bigsqcup_{g \in G \cap C} g + V \;\subseteq\; C + V \;\subseteq\; C + C.$$

We assumed that $C + C$ has finite measure and monotonicity of μ gives the contradiction: V is not μ-measurable. □

Theorem 2.3 might have gained some generality, but introduced many more conditions. In the special case $X = \mathbb{R}$, it can however be greatly simplified, as a result of the following lemma[2]:

LEMMA 2.4. *Let G be a subgroup of $\mathbb{R}$. There exists a bounded set C containing infinitely many elements from G if and only if G is dense in $\mathbb{R}$.*

Proof. The right to left implication is clear. Conversely, let x be a real number and $g > x$ be some fixed element of G. We have to show that every open neighbourhood has a nonempty intersection with G. Let $\delta > 0$ be given and define

$$\varepsilon = \min(\delta, d), \qquad \text{where } d = d(g, B_x(\delta)) = |g - (x + \delta)|$$

Since C is bounded and contains infinitely many points of G, we can find elements $h > h' \in G$ such that $0 < h - h' < \varepsilon$. Otherwise a lower bound on the distance between points in $G \cap C$ would exist and only finitely many points would fit in C. Now repeatedly subtract $h - h'$ from x until we reach $B_x(\delta)$ while remaining inside G. More precisely, there exists an $n \in \mathbb{N}$ such that

$$g - n(h - h') \in B_x(\delta) \cap G,$$

since G is closed under addition. This finishes the proof. □

[2] More generally, it is easily seen that LEMMA 2.4 is true in any metric space with the property that every bounded sequence has a Cauchy subsequence. These are precisely the totally bounded metric spaces.

Now we can describe the non-measurable sets generated by Vitali's construction.

DEFINITION 2.5. Let G be a countable, dense subgroup of $\mathbb{R}$ with addition. A selector V for $\mathbb{R}/G$ is called a *G-Vitali set*. If we do not want to specify the subgroup, we speak of a *Vitali-type* set.

THEOREM 2.6. *If μ is a translation invariant measure on $\mathbb{R}$, no G-Vitali set is μ-measurable. In particular no G-Vitali set is Lebesgue-measurable.*

Proof. By definition, G is understood to be a dense additive subgroup of R. Let V be a selector of $\mathbb{R}/G$ and C a bounded set containing infinitely many points, which exists by LEMMA 2.4. Consider two cases. If V is bounded then we can find an interval I containing V and C. Then $m(I+I)$ is finite, so we may apply Theorem 2.3 to prove the result.

If V is not bounded, follow the proof of Theorem 2.3 to conclude that $m(V) > 0$. Then there must exist an n such that $m(V \cap [-n,n]) > 0$. Notice how

$$m\Big(\bigsqcup_{g\in G\cap C} g + (V\cap[-n,n])\Big) = \sum_{g\in G\cap C} m(V\cap[-n,n]) = \infty,$$

while this union is also included in $C+[-n,n]$, which must have finite measure. This gives a contradiction and shows that V is not measurable. □

Example 2.7 (A Vitali-type subset of the circle)**.** After the abstract treatment of Vitali sets, a more visual example seems appropriate; therefore consider the circle group $\mathbb{T}$. It is most convenient to interpret this group as the unit circle in the complex plane with multiplication as group operation: $e^{i\varphi}\cdot e^{i\psi} = e^{i(\varphi+\psi)}$. We can further identify $\mathbb{T}$ with the interval $[-\pi,\pi)$ by mapping points to their arguments and transfer the Lebesgue measure from $\mathbb{R}$ to $\mathbb{T}$: define the measure $\mu(A)$ to be $m(\{\arg(z) : z \in A\})$ if the latter exists.

The set

$$\mathbb{T}_{\mathbb{Q}} := \{e^{iq} \in \mathbb{T} : q \in [-\pi,\pi)\cap\mathbb{Q}\}$$

of points with rational arguments is easily seen to form a dense subgroup. Moreover μ is invariant under $\mathbb{T}_{\mathbb{Q}}$: A set $A \subseteq \mathbb{T}$ is measurable set if and only if $\{\arg(z) : z \in A\} \subseteq \mathbb{R}$ is measurable, which will remain measurable under translation by a rational q. In other words, $e^{iq}\cdot A = \{e^{iq}\cdot z : z \in A\}$ is measurable. Since the second condition in Theorem 2.3 is always satisfied by taking $C = \mathbb{T}$ we may apply the theorem to conclude that any selector of $\mathbb{T}/\mathbb{T}_{\mathbb{Q}}$ is non-measurable.

It might not be entirely clear why this is in fact a very *visual* example, so let me rephrase the construction. If we start with any point z on the unit circle, we could consider all the points reached by rational rotations of z. These points form the the *orbit* of z under rational rotation, which coincides with the equivalence class $[z]$ in $\mathbb{T}/\mathbb{T}_{\mathbb{Q}}$. The collection of all orbits partitions $\mathbb{T}$ and all of its selectors are non-measurable. Such a selector is thus a set containing only points of the circle that differ by a irrational angle.

2.2. Non-measurable selectors of non-dense subgroups

So far we have discussed countable dense subgroups, but what happens if $G \neq \{0\}$ is not a dense subgroup of $\mathbb{R}$? We investigate this for the Lebesgue measure. By LEMMA 2.4, a bounded interval $[0, x]$ contains finitely many points of G, which thus has a smallest positive element[3] g that generates the group: If not, there would exist an h in G such that $h \neq n \cdot g$ for all $n \in \mathbb{Z}$. Let M be the greatest positive integer such that $Mg < h < (M+1)g$. After subtracting Mg from all sides we see that $h - Mg$ is a positive element of G strictly smaller than g; a contradiction. This shows that the non-dense subgroups[4] G are of the form

$$G = g\mathbb{Z} = \{\dots, -2g, -g, 0, g, 2g, \dots\}.$$

It is clear that $\mathbb{R}/G$ has a measurable selector: the interval $[0, g)$. This is a selector because any point $x \in \mathbb{R}$ can be written as $ng + \delta$ with $0 \leq \delta < g$ and $n \in \mathbb{Z}$, such that $x - \delta = ng \in G$ or equivalently, $x \sim \delta$. The selector $[0, g)$ has positive measure and this is true for any *measurable* selector S of $\mathbb{R}/G$: write $\mathbb{R}$ as the disjoint union $\bigsqcup_{n\in\mathbb{Z}} S + ng$ and apply countable additivity. The following question rises: are all selectors measurable? The answer is negative and to prove this we will construct a non-measurable selector. We will use a result obtained in the next chapter, namely that every measurable set (of positive measure) contains a non-measurable set. In particular we use that this non-measurable is *saturated*:

DEFINITION 2.8. A set $E \subseteq \mathbb{R}$ is called *saturated non-measurable* if for every Lebesgue measurable subset A of $\mathbb{R}$ the following holds:

$$m_*(E \cap A) = 0 \quad \text{and} \quad m^*(E \cap A) = m(A).$$

Alternatively, saturated non-measurable sets can be given the following characterisations.

THEOREM 2.9. *A set $E \subseteq \mathbb{R}$ is saturated non-measurable if and only if any of the following conditions hold:*

(i) $m_*(E) = 0 = m_*(E^c)$.

(ii) $A \cap E$ *is non-measurable for every measurable set* $A \subseteq \mathbb{R}$ *of positive measure.*

Proof. See Appendix A, Theorem A.6. □

Saturated non-measurable sets exhibit a rather extreme form of nonmeasurability, in the sense that they maximize the difference between the inner and outer measure. For measurable sets, those should coincide by Theorem 1.5. For example, taking $A = \mathbb{R}$, a saturated non-measurable set is too small to contain a set of positive measure, yet is so

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BIBLIOGRAPHIC NOTE. Saturated non-measurable sets are inspired on Sardella and Ziliotti 2002 and Armstrong 2010

[3] The non-existence of a smallest positive element is another characterisation of a dense subgroup.

[4] If G is a nondense subgroup, hence of the form $g\mathbb{Z}$, then its complement is the countable union of open intervals $(ng, (n+1)g)$ for $n \in \mathbb{Z}$. It follows that G is a closed subset of $\mathbb{R}$, classifying all subgroups in a topological sense: they are either dense or closed.

large that every cover of measurable sets necessarily covers $\mathbb{R}$. Do such sets exits at all? Yes, in the next chapter, Bernstein sets turn out to be very natural examples. They exist under the assumption of $\mathsf{WO}(\mathbb{R})$. I refer to Appendix A, Theorem A.8 for a construction of a saturated non-measurable set based on a Vitali-type set, also using $\mathsf{WO}(\mathbb{R})$. For now, we will assume that every set of positive measure contains a saturated non-measurable set — the content of PROPOSITION 3.10 — in the construction of a non-measurable selector for $\mathbb{R}/g\mathbb{Z}$.

Our starting point is the selector $[0, g)$ and its saturated non-measurable subset B. Consider the selector

$$S := ([0, g) \cap B^c) \sqcup (B + g),$$

which is a subset of $[0, 2g)$. We show that S is saturated non-measurable by showing $m_*(S) = 0 = m_*([0, 2g) \setminus S)$. So let F be a closed subset of S and consider

$$F_1 := F \cap [0, g] \quad \text{and} \quad F_2 := F \cap [g, 2g].$$

These are disjoint subsets of $[0, g) \cap B^c$ and $B + g$ respectively since they are inside S and cannot both contain g. It is also clear that F_1 and F_2 are closed. Since any closed set contained in the saturated non-measurable set B^c has measure zero, we find $m_*(F_1) = 0$. Similarly, $F_2 - g$ is a closed subset of B so F_2 is a null set and we find that

$$m(F) = m(F_1 \sqcup F_2) = m(F_1) + m(F_2) = 0.$$

As a result $m_*(S) = 0$.

Now let F be a closed subset of $[0, 2g) \setminus S = B \cup ([g, 2g) \cap (B^c + g))$ and again split it in the same two parts F_1 and F_2. As F_1 is a closed subset of B, and $F_2 - g$ a closed subset of B^c, both F_1 and F_2 have measure zero. The conclusion $m_*([0, 2g) \setminus S) = 0$ follows and we have shown that S is saturated non-measurable. Theorem 2.10 summarizes this result.

THEOREM 2.10. *Assume $\mathsf{WO}(\mathbb{R})$ and let G be a countable subgroup of $\mathbb{R}$. Then G is dense in $\mathbb{R}$ if and only if all selectors of $\mathbb{R}/G$ are non-Lebesgue measurable. If G is not dense, it has both measurable and non-measurable selectors.*

Proof. If G is dense, Theorem 2.6 ensures that all selectors are non-measurable. Conversely, suppose that G has only non-measurable selectors and is not dense. As seen earlier, G should be of the form $g\mathbb{Z}$; this lead to a contradiction as $[0, g)$ is a measurable selector of $\mathbb{R}/g\mathbb{Z}$. Furthermore, if $G = g\mathbb{Z}$ is not dense then $[0, g)$ is a measurable selector and a non-measurable one is given in the above construction. □

2.3. Non-measurable sets from Hamel bases

Vitali used a well-ordering on $\mathbb{R}$ to obtain a selector for $\mathbb{R}/\mathbb{Q}$. In this section a different way to obtain a Vitali-type set is discussed. The starting point is Theorem 1.8, which shows that AC implies the existence of a basis for any vector space. In infinite dimensional spaces these bases are often referred to as *Hamel bases* to distinguish them from other notions (such as Schauder bases).

DEFINITION 2.11. Let V be a vector space over a field $\mathbb{F}$ and $\mathcal{B} \subseteq V$. If the following two conditions are satisfied, $\mathcal{B}$ is called a *Hamel basis of V*.

(i) $\mathcal{B}$ is linearly independent: if f $\alpha_1 b_1 + \cdots + \alpha_n b_n = 0$ for basis vectors $b_i \in \mathcal{B}$ and scalars α_i, then $\alpha_1 = \cdots = \alpha_n = 0$; and

(ii) $\mathcal{B}$ spans the whole space, i.e. $\operatorname{Span} \mathcal{B} = V$.

A very interesting example is $\mathbb{R}$, considered as a vector space over $\mathbb{Q}$. Less then full AC is needed to proof the existence of a Hamel basis:

THEOREM 2.12. $\mathsf{ZF} \vdash (\mathsf{WO}(\mathbb{R}) \longrightarrow \mathsf{VSB}_{\mathbb{Q}}(\mathbb{R}))$

Proof. Now follow the proof of Theorem 1.8, replacing Zorn's Lemma by an induction argument. Write $\mathbb{R} = \{x_\alpha : \alpha \leq 2^{\aleph_0}\}$ and then recursively define a function $\mathcal{B}$ by $\mathcal{B}(x_0) := \{x_0\}$ and

$$\mathcal{B}(\alpha) := \begin{cases} \bigcup\{\mathcal{B}(\gamma) : \gamma < \alpha\} \cup \{x_\alpha\} & \text{if this is linearly independent.} \\ \bigcup\{\mathcal{B}(\gamma) : \gamma < \alpha\} & \text{otherwise} \end{cases}$$

We claim that $\mathcal{H} := \mathcal{B}(2^{\aleph_0})$ is a Hamel basis for $\mathbb{R}$ and prove linear independence inductively (we do not need to handle successors and limits seperately). Suppose that for some α, all $\mathcal{B}(\gamma)$ are independent for $\gamma < \alpha$. If $\bigcup_{\gamma<\alpha} \mathcal{B}(\gamma)$ contains a finite, linearly dependent set X, then $X \subseteq \mathcal{B}(\gamma)$ for some $\gamma < \alpha$, contradicting $\mathcal{B}(\gamma)$'s linear independence. So $\mathcal{B}(\alpha)$ is independent, hence $\mathcal{H}$ by induction.

Next we claim that $\operatorname{Span} \mathcal{H} = \mathbb{R}$. If not, then there exists x_α not in $\operatorname{Span} \mathcal{B}(\beta)$ for all $\beta < 2^{\aleph_0}$. In particular, it is not in $\operatorname{Span} \mathcal{B}(\gamma)$ for all $\gamma < \alpha$. But then every linear independent subset of $\mathcal{B}(\alpha)$ remains independent when x_α is added, hence $x_\alpha \in \mathcal{B}(\alpha)$ by construction. This is a contradiction, so $\mathcal{H}$ spans $\mathbb{R}$ and is thus a Hamel basis. □

It is unknown (to the author) if the converse implication is also true: does the existence of a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$ implies that $\mathbb{R}$ is well-orderable? If not, the following non-measurable set uses an ever weaker version of choice than the previous constructions.

THEOREM 2.13. *Assume that $\mathbb{R}$ over $\mathbb{Q}$ has a Hamel basis, i.e. $\mathsf{VSB}_{\mathbb{Q}}(\mathbb{R})$. Then there exists a Vitali-type non-measurable set.*[5]

Proof. Let $\mathcal{B}$ be a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$ and $b \in \mathcal{B}$ a fixed element. We claim that $V = \operatorname{Span} \mathcal{B} \setminus \{b\}$ is a G-Vitali set, hence not Lebesgue measurable, for $G = b \cdot \mathbb{Q}$. Here $b \cdot \mathbb{Q}$ is an additive — not a multiplicative — subgroup of $\mathbb{R}$. It is easily seen to be dense, so we show that V gives a system of representatives for $\mathbb{R}/b \cdot \mathbb{Q}$.

Fix a point $x = \alpha_1 b_1 + \cdots + \alpha_n b_n$ in $\mathbb{R}$, where $b_i \in \mathcal{B}$ are basis vectors and $\alpha_i \in \mathbb{Q}$ scalars. Either none of the b_i's equals b — in which case x is in V and we're done — or one of them equals b. Assume $b_i = b$. Reindexing yields

$$x = \left(\sum_{k=1}^{n-1} \alpha_k b_k\right) + \alpha_i b_i \quad \text{hence} \quad v = \sum_{k=1}^{n-1} \alpha_k b_k \in V$$

[5]Based on Kharazishvili 2004, p35-36.

is the element in V equivalent to x. This shows that V is a selector for $\mathbb{R}/b \cdot \mathbb{Q}$ and Theorem 2.6 proves that V is not Lebesgue measurable. □

Conclusion Using the generalisation of Vitali's construction, many non-measurable sets can found on the assumption of $\mathsf{WO}(\mathbb{R})$: every dense subgroup has many different selectors of its quotient group and all those will not be measurable. The weakest assumption needed so far to construct such a Vitali-type set is $\mathsf{VSB}_{\mathbb{Q}}(\mathbb{R})$: the existence of a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$.

As observed in PROPOSITION 2.2, extensions of the Lebesgue measures can not be translation invariant. This vividly demonstrates that the Vitali construction crucially relies on algebraic properties of the Lebesgue measure. Of course, it has many more, and in the next chapter we will see how an essentially topological property, regularity, can give a non-measurable set.

CHAPTER 3

Bernstein Sets

As anticipated, this chapter will explore how topological properties of the Lebesgue measure lead to non-measurability of certain sets discovered by Bernstein. The construction is based on the observation that the measure of a measurable set A can be approximated from below by the measures of closed and bounded sets lying inside A — this is the *inner regularity* of the measure. The clever insight is the following: if we can split $\mathbb{R}$ into two parts such that they both *only* contain countable closed subsets, then both parts can only be approximated by null sets. As a result these two parts cannot be measurable, since that would mean that $\mathbb{R}$ has measure zero.

Slightly rephrasing this approach, we look for a set B such that B and its complement B^c both intersect all uncountable closed subsets (and thus contain none). In $\mathbb{R}$ this property coincides with the more general notion of a *Bernstein set*. We will introduce *Bernstein sets* in their usual formulation[1], work towards the simpler formulation[2] on $\mathbb{R}$ and then demonstrate their existence, which relies on a well-ordering of $\mathbb{R}$.

BIBLIOGRAPHIC NOTE. This chapter mainly uses ideas put forward in Oxtoby (1980), Reimann (2011) and Kharazishvili (2004).

[1] cf. Kharazishvili 2004.

[2] cf. Oxtoby 1980.

3.1. Perfect sets and uncountable closed sets

The usual definition of Bernstein sets uses the notion of a *perfect set*: a closed set without isolated points. In a topological space X with a subset A, a point $a \in A$ is called an *isolated point of A* if it can be separated from all other points of A by some open set $U \subseteq X$, i.e. if $U \cap A = \{a\}$. As an example any finite set in $\mathbb{R}$ could be considered. It consists only of isolated points and has no perfect subsets. However, the complement of a finite set does contain a perfect set: an closed interval, for example. For Bernstein sets, we would want neither the set, nor its complement to contain nonempty perfect subsets:

DEFINITION 3.1. A set $B \subseteq X$ is called *Bernstein* if both B and B^c contain no nonempty perfect subsets. That is, if for each nonempty perfect set $P \subseteq X$ we have

$$P \cap B \neq \emptyset \quad \text{and} \quad P \cap B^c \neq \emptyset.$$

As noted above, Bernstein subsets of $\mathbb{R}$ can be more easily characterised in terms of uncountable closed sets. We show that nonempty perfect sets and uncountable closed sets are closely related: perfect sets are uncountable and conversely, every uncountable set contains a perfect set.

Starting point is the *Cantor set*. It is constructed by sucessively removing the middle thirds from $[0, 1]$. More precisely, start with $C_0 = [0, 1]$ and let C_{n+1} be the union of all intervals in C_n, without their middle thirds. This is illustrated in Figure 3.1. Then we define the Cantor set C to be $\bigcap_{n\in\mathbb{N}} C_n$. It has many interesting properties. For instance, the total length C_n is easily seen to be $(\frac{2}{3})^n$ and as a result, C is a null set. But it is also uncountable. To see that, we could observe that x is in C if and only if its ternary expansion

$$x = \sum_{k\in\mathbb{N}} \frac{x_k}{3^k} = 0.x_1x_2x_3x_4\ldots$$

contains only 0's and 2's, i.e. $x_k \in \{0, 2\}$. After all, $x \in C_n$ if and only if $x_k \in \{0, 2\}$ for all $k \leq n$, as can be shown by a simple induction. Therefore, $x \in C$ if and only if $x_k \in \{0, 2\}$ for all $k \in \mathbb{N}$. Now replace the 2's by 1's in the ternary expansion to bijectively map elements in C to points in $[0, 1]$, written in their binary expansion. This shows $C \sim [0, 1] \sim \mathbb{R}$.

But wait, can we be sure that C is nonempty? Well, not yet; it is ensured by the following lemma.

LEMMA 3.2 (CANTOR'S INTERSECTION THEOREM). *Let $\{C_n : n \in \mathbb{N}\}$ be a decreasing sequence of closed subsets of $\mathbb{R}$, i.e. $C_{n+1} \subseteq C_n$ for all $n \in \mathbb{N}$, whose diameter*

$$\operatorname{diam}(C_n) := \sup\{|x - y| : x, y \in C_n\},$$

tends to zero. Then the intersection $C := \bigcap_{n\in\mathbb{N}} C_n$ contains precisely one point.

Proof. Clearly $\operatorname{diam}(C) = 0$, so either C is empty or it contains one point. To see it is nonempty, select an $x_n \in C_n$ for each $n \in \mathbb{N}$. We do not need choice for this, since we could fix a point $x^\star$ and choose the point in C_n closest to that. This exists since C_n is closed. The sequence $\{x_n : n \in \mathbb{N}\}$ thus obtained, must be Cauchy since for any $n < m$ we have

Figure 3.1.: The first 5 steps in the construction of the Cantor set. In each step C_{n+1} is constructed by removing the middle thirds of the intervals in C_n. In this way, C_n consists of 2^n interval, each of length $(\frac{1}{3})^n$ and as a result $m(C_n) = (\frac{2}{3})^n$. The Cantor set is defined as the intersection of all these sets: $C := \bigcap_{n\in\mathbb{N}} C_n$.

$|x_n - x_m| \leq \text{diam}(C_n)$, which tends to zero. As $\mathbb{R}$ is complete, the sequence converges to a unique limit x. This is moreover a limit point of each C_n, that is closed, so $x \in C_n$ for all n and the proof is complete. □

Cantor's construction can be extended to arbitrary perfect sets, to prove the following theorem.

THEOREM 3.3 (CANTOR). *Every nonempty perfect set $P \subseteq \mathbb{R}$ contains an uncountable subset. Consequently, P is in bijection to $\mathbb{R}$.*

Proof. Since P is perfect and nonempty, we can choose two distinct points in P. Around them, we can find closed intervals, small enough to ensure that they are disjoint and of length at most $1/3$. Denote those two intervals by $I(0)$ and $I(1)$. We recursively define a sequence of decreasing subintervals.

Suppose that 2^n disjoint closed intervals, each meeting P, have been constructed. We denote these intervals by $I(i_1, \ldots i_n)$ where $\mathbf{i} := i_1, \ldots, i_k$ is a sequence of 0's and 1's. Because any such interval $I(\mathbf{i})$ meets P, we can find two points x and y satisfying some criteria. They may not be endpoints of $I(\mathbf{i})$ and must both be in P (which is possible, since it is perfect). Then we can find an $\varepsilon > 0$ such that

$$\varepsilon < \frac{1}{3^{n+1}}, \qquad \text{and} \qquad \varepsilon < |x - y|.$$

Moreover, ε must smaller than the smallest distance from x or y to an endpoint of $I(\mathbf{i})$. This all ensures that the closed intervals of radius ε around x and y are disjoint and completely inside $I(\mathbf{i})$. Name those two intervals $I(i_1, \ldots, i_n, 0)$ and $I(i_1, \ldots, i_n, 1)$. Again, these intervals intersect P.

Now we claim that the set

$$C := \bigcap_{n=1}^{\infty} \bigcup_{i_1,\ldots,i_n} I(i_1, \ldots, i_n). \tag{3.1}$$

is in bijection to $\mathbb{R}$. Suppose that $x \in C$. Then it corresponds to a sequence $\{i_n\}$ of zeros and ones: simply let $i_{n+1} := 1$ if $x \in I(i_1, \ldots, i_n, 1)$ and $i_{n+1} := 0$ otherwise. This sequence is unique for each x, since the intervals are ensured to be disjoint at each step. But the converse also holds. If $\{i_n : n \in \mathbb{N}\}$ is a sequence of zero's and ones, then the corresponding sequence defined by $I_n := I(i_1, \ldots, i_n)$ is decreasing and consists of closed sets, whose diameter tends to zero by construction. It follows from LEMMA 3.2 that $\bigcap_{n\in\mathbb{N}} I_n$ contains a unique point in C. This establishes a bijection between C and the zero-one sequences in ${}^{\mathbb{N}}2$, which was known to be equinumerous to $\mathbb{R}$. So $C \sim \mathbb{R}$. □

THEOREM 3.4 (CANTOR-BENDIXON). *Every uncountable closed subset of $\mathbb{R}$ contains a perfect set. Consequently, any closed set is either countable or has the cardinality of the continuum.*

Proof. Let F be a uncountable closed subset of $\mathbb{R}$. We will show that the set C of condensation points is perfect. A *condensation point of F* is a point for which every neighbourhood contains uncountably many points of F. C is closed because surely, all its points are accumulation points. It suffices to show that $B_x(\varepsilon) \cap C$ is uncountable for all $x \in C$ and $\varepsilon > 0$, since this in particular shows that C has no isolated points. Therefore, we prove that $F \setminus C$ is countable.

Let $z \in F \setminus C$. It follows from the definition of C, that there must be some neighbourhood U_z of z such that $U_z \cap F$ is (only) countable. Without loss of generality, we may furthermore assume U_z to be an interval with rational endpoints, of which there are countably many. This shows that $F \setminus C$ can be covered with countably many rational intervals, each containing countably many points in C. We conclude that $F \setminus C$ is countable and thus finish the proof. □

It is interesting to note that this answers part of the question asked by the (weak) Continuum Hypothesis: does every uncountable subset of $\mathbb{R}$ has the same cardinality as $\mathbb{R}$? Apparently, this is true for all closed sets. Let's now return to the of our interest: Bernstein sets in $\mathbb{R}$.

3.2. Bernstein subsets of the real line

THEOREM 3.5. *A subset B of $\mathbb{R}$ is Bernstein if and only if both B and B^c intersect every closed uncountable subset of $\mathbb{R}$.*

Proof. Let B be Bernstein set and F any closed uncountable subset of $\mathbb{R}$. By Theorem 3.4 F contains a perfect subset P. It follows directly that both B and B^c intersect F. Conversely, if A and A^c intersect all closed uncountable subsets, they certainly intersect all perfect subsets of $\mathbb{R}$ since they are uncountable by Theorem 3.3. Thus A is Bernstein. □

So far we have established a more straightforward way to identify Bernstein sets. Convenient as this may be, their very existence is still uncertain. But not for long; first we need a lemma.

LEMMA 3.6. *There exist continuum many uncountable closed subsets of $\mathbb{R}$.*

Proof. Let Q be the countable collection of open intervals with rational endpoints. Every open set U can be represented as a union of some subset Q' of these intervals: $U = \bigcup Q'$. This establishes a correspondence between $\mathcal{P}(Q)$ and the open intervals. Since $\mathcal{P}(Q) \sim \mathcal{P}(\mathbb{N}) \sim \mathbb{R}$, it follows that there are at most continuum many open sets and by complementation equally many closed sets. There are, however, at least continuum many closed intervals, since each interval $[a, b]$ corresponds uniquely to a point $(a, b) \in \mathbb{R}^2 \setminus D$ where D denotes the diagonal. This is easily seen [3] to be equinumerous to $\mathbb{R}$, hence there are exactly continuum many uncountable closed subsets of $\mathbb{R}$. □

THEOREM 3.7. *Assuming* WO($\mathbb{R}$), *there exists a Bernstein subset of* $\mathbb{R}$.

Proof. Let $\mathcal{F}$ denote the class of uncountable closed subsets of $\mathbb{R}$. By the well-ordering or $\mathbb{R}$, let p_F and q_F denote the smallest two elements of $F \in \mathcal{F}$. The set B containing all p_F will intersect each uncountable closed set, and so will its complement as it contains all q_F. Thus B is Bernstein.

Fleshing out this argument, write $\mathcal{F} = \{F_\alpha : \alpha < 2^{\aleph_0}\}$ using LEMMA 3.6. We recursively define p_α and q_α for all $\alpha < 2^{\aleph_0}$. Let p_1 and q_1 then be the first two (distinct) members of F_1. Now consider $1 < \alpha < 2^{\aleph_0}$ and suppose p_β and q_β have been defined for all $\beta < \alpha$. Then define p_α and q_α to be the first two distinct elements of

$$F_\alpha \setminus \Big(\bigcup_{\beta<\alpha} \{p_\beta, q_\beta\} \Big). \tag{3.2}$$

The points p_α and q_α are only defined if the above set is nonempty. That is indeed the case since F_α has cardinality $2^{\aleph_0}$ and the union in (3.2) has strictly smaller cardinality $2 \cdot \mathrm{Card}(\alpha)$: this is finite if α is and equals $\mathrm{Card}(\alpha) < 2^{\aleph_0}$ otherwise. We have thus picked two unique elements from each F_α and can define the Bernstein set $B = \{p_\alpha \in \mathbb{R} : \alpha < 2^{\aleph_0}\}$. Indeed both B and B^c intersect each closed uncountable set F_α since $p_\alpha \in B \cap F_\alpha$ and $q_\alpha \in B^c \cap F_\alpha$. □

3.3. Measuring Bernstein sets

The reason for the nonmeasurability of Bernstein sets was mentioned in the introduction, but we look closer into it.

THEOREM 3.8. *Let μ be an inner regular diffuse measure on $\mathbb{R}$. Every μ-measurable subset of a Berstein set B has measure zero. As a result, B and B^c are both not μ-measurable.*

Proof. Let $B \subseteq \mathbb{R}$ be a Bernstein set and recall from page 1.2 that inner regular measures satisfy

$$\mu(A) = \sup\{\mu(F) : \ F \subseteq A \text{ is a closed set}\}.$$

[3] Clearly, $\mathbb{R}^2 \setminus D$ injects into $\mathbb{R}^2$, which happens to be equinumerous to $\mathbb{R}$. An elementary proof of this, due to Julius König, interweaves the decimal expansions of two points x and y to obtain a uniquely determined point z; see for example Aigner and Ziegler (2010, p. 109) for details. Next, one can inject $\mathbb{R}$ into $\mathbb{R}^2 \setminus D$: by $0 \mapsto (1, 0)$, $\frac{1}{n} \mapsto (\frac{1}{n+1}, 0)$ and $x \mapsto (x, 0)$ for all other x.

Any closed set $F \subseteq A$ must be countable, otherwise it would intersect B^c and not be inside A, by definition of a Bernstein sets. Since μ is a diffuse measure, $\mu(A) = 0$. Next suppose B, hence B^c, is measurable. The above shows that they must both be null sets, which implies that $\mathbb{R} = B \sqcup B^c$ has measure zero, a contradiction with $\mu(\mathbb{R}) > 0$. □

This theorem has several direct consequences.

PROPOSITION 3.9. *If μ is an extension of the Lebesgue measure to $\mathcal{P}(\mathbb{R})$, then μ can not be inner regular.*

Now consider any set A of positive μ-measure and a Bernstein set B. The same arguments show that $B \cap A$ cannot be measurable, since this would partition A into two nullsets, contradicting that $\mu(A) > 0$. We have found the following result.

PROPOSITION 3.10. *Assume $\mathsf{WO}(\mathbb{R})$ and let μ be an inner regular diffuse measure on $\mathbb{R}$. Any set of positive μ-measure has a non-measurable subset.*

Remember that Theorem 2.9 identified saturated non-measurable sets as the ones that have a non-measurable intersection with any set of positive measure. As a direct consequence of PROPOSITION 3.10, we thus find

PROPOSITION 3.11. *Bernstein sets are saturated non-measurable sets.*

Conclusion Where Vitali's construction took advantage of algebraic properties of measures to find a non-measurable set, Bernstein exclusively used topological properties of the measure and $\mathbb{R}$. Assuming $\mathsf{WO}(\mathbb{R})$, we showed that Bernstein sets exist. No inner-regular weak measure can measure these sets, and in particular the Lebesgue measure can't. It is no surprise that properties of $\mathbb{R}$ can determine the existence of non-measurable sets — this does not depend exclusively on the measure. In the next chapter we continue investigations to set-theoretic properties of $\mathbb{R}$ that ensure non-measurable sets.

CHAPTER 4

Fubini and Sierpiński's Partition

The nonmeasurability of Bernstein sets was not difficult to establish; the difficulty lay in their existence. The same is true for the construction presented in this chapter. Here, we leave the real line and head for the Euclidean plane. The product measure defined there essentially sums up the measures of all 'slices' — *sections* — of a set. It is intuitively clear that if all of those sections have measure zero, then the set should have measure zero as well. This intuition is indeed correct and suggests that we should try to partition the plane into two parts, both built up from countable sections. Were we to succeed, the plane would the union of two nullsets, which is absurd. Rather, we conclude that the sets in the partition cannot be measurable.

It was the Polish mathematician Sierpiński who first observed that $\omega_1 \times \omega_1$ could be partitioned in the desired way, i.e. into two parts with only countable sections. Therefore, the existence of such a partition of $\mathbb{R} \times \mathbb{R}$ becomes dependent on the Continuum Hypothesis. We explore all of this below in more detail.

4.1. Product measures

If we start with two σ-finite measure spaces (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) we can turn these into a product space (X, Σ, μ) where X is the cartesian product of X_1 and X_2 and Σ and μ still have to be defined. The most straightforward way is to start with the 'rectangles'

$$\mathcal{R} = \{A_1 \times A_2 : A_1 \in \Sigma_1 \text{ and } A_2 \in \Sigma_2\}$$

Whatever the product measure will be, rectangles $A_1 \times A_2$ should have measure $\mu_1(A_1)\mu_2(A_2)$. General sets $A \subset X$ are not rectangular, so we proceed by 'slicing' A into measurable sections, whose measures we 'add up' to obtain the total measure. To state this formally we need the notion of a *section* of a subset A. There are two variants:

$$A_x := \{y \in X_2 : (x, y) \in A\} \quad \text{and} \quad A^y := \{x \in X_1 : (x, y) \in A\}.$$

Whenever A is in the product σ-field Σ, both sections will be measurable for every choice of x and y respectively. That is: $A_x \in \mathcal{F}_2$ for every $x \in X_1$ and $A^y \in \mathcal{F}_1$ for every $y \in \mathcal{F}_2$. It thus makes sense to write $\mu_2(A_x)$ and $\mu_1(A^y)$ and we will define the *product measure* $\mu = \mu_1 \times \mu_2$ for $A \in \Sigma$ by

$$\mu(A) := \int_{X_1} \mu_2(A_x)\, \mathrm{d}\mu_1 = \int_{X_2} \mu_1(A^y)\, \mathrm{d}\mu_2. \tag{4.1}$$

Here we assumed that the functions $y \mapsto \mu_1(A^y)$ and $x \mapsto \mu_2(A_x)$ are measurable, and that the integrals equal, which is of course the case.[1] Now we can formulate a crucial result related to Fubini's classical Theorem.

THEOREM 4.1 (FUBINI). *Let $A \in \Sigma$ be measurable and presume that one of the sets*

$$B_1 := \{x \in X_1 : \mu_2(A_x) > 0\} \quad \textit{or} \quad B_2 := \{y \in X_2 : \mu_1(A^y) > 0\}$$

has measure zero with respect to μ_1 or μ_2 respectively. Then $\mu(A) = 0$.

Proof. This follows directly from the definition of the product measure. Presume that $\mu_1(B_1) = 0$; the case for B_2 is similar. Then

$$\mu(A) = \int_{X_1} \mu_2(A_x)\, \mathrm{d}\mu_1 = \int_{B_1} \mu_2(A_x)\, \mathrm{d}\mu_1 = 0.$$

The second equality holds since $\mu_2(A_x)$ is zero outside of B_1. Integration over null sets gives integral 0, proving the third equality. □

4.2. The Sierpínski partition

In this section we work towards the non-measurable *Sierpiński partition* of the Euclidean plane. At the heart of the construction lies a certain linear order $<$ with the property that every proper initial segment $\mathrm{IS}(x)$ is countable.

[1] e.g. Capiński and Kopp 2004, Theorem 6.5 and 6.9.

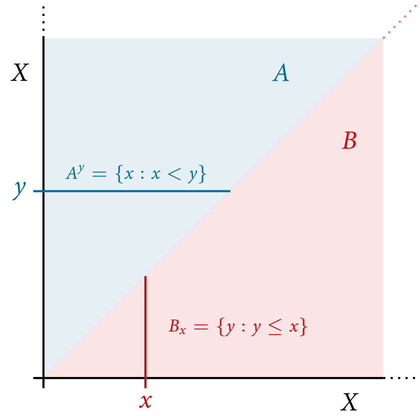

Figure 4.1.: The Sierpiński partition $\{A, B\}$ of the product set $X \times X$. Since $(X, <)$ is an initially countable ordering, all of the sections A^y and B_x are countable. Graphically, they can be thought of as the horizontal and vertical line respectively. Note that this is slightly imprecise as the sections are not subsets of $X \times X$ but of X.

DEFINITION 4.2. If $(X, <)$ is an uncountable, linearly ordered set with the property that $\mathrm{IS}(x)$ is countable for all $x \in X$. Then we call $(X, <)$ an *initially countable ordering*. It induces the *Sierpiński partition* $\{A, B\}$ of the product set $X \times X$:

$$A = \{(x, y) : x < y\} \quad \text{and} \quad B = \{(x, y) : y \leq x\}. \tag{4.2}$$

If $\{A, B\}$ is a Sierpiński partition, then for all $x, y \in X$, the sections

$$A^y = \{x : (x, y) \in A\} = \mathrm{IS}(y) \quad \text{and} \quad B_x = \{y : (x, y) \in B\} = \overline{\mathrm{IS}}(x)$$

are countable. This is illustrated in Figure 4.1. Whenever X is equipped with a σ-finite measure μ, these sections must be of measure zero. The set of points y where A^y has positive measure, is then empty, hence of measure zero. The same goes for $\{x : \mu(B_x) > 0\}$. At this point, we can turn to Theorem 4.1 to see that if A or B would be measurable, they would both have measure zero with respect to the product measure $\mu \times \mu$. Consequently $X \times X$ has to be a null set, contradicting the fact that a measure is nonzero. We have thus proved the following theorem.

THEOREM 4.3. *If X is a space equipped with a σ-finite measure and a initially countable ordering, then the sets in the corresponding Sierpiński partition of $X \times X$ are non-measurable with respect to the product measure.*

The canonical example of such a space X is the first uncountable ordinal, ω_1. By definition, each initial segment of this ordinal is countable. The strong version of the Continuum Hypothesis — we will later look at the details — says that ω_1 can be bijectively mapped to $\mathbb{R}$, in that way inducing its initially countable ordering on $\mathbb{R}$. As a result we can find a Sierpiński partition $\{A, B\}$ of $\mathbb{R}^2$, which consists of nonmeasurable sets with respect to the Lebesgue measure on $\mathbb{R}^2$ by Theorem 4.3. Let us now more carefully look at initially countable orderings and their relation to the Continuum Hypothesis.

Speaking of *the* Continuum Hypothesis, obscures the fact that it has different formulations:

CH (Strong Continuum Hypothesis.) $\mathbb{R} \sim \omega_1$.

wCH (Weak Continuum Hypothesis.) If $A \subseteq \mathbb{R}$ and $A \not\sim \mathbb{N}$, then $A \sim \mathbb{R}$.

Indeed, wCH is readily implied by CH. And when some choice is allowed, they are even equivalent:

THEOREM 4.4. $\mathsf{ZF} + \mathsf{WO}(\mathbb{R}) \vdash (\mathsf{CH} \longleftrightarrow \mathsf{wCH})$.

Proof. We only prove that wCH implies CH. By $\mathsf{WO}(\mathbb{R})$ there exists a well-ordering of $\mathbb{R}$. The fundamental theorem of well-orders, gives that either $\mathbb{R} \sqsubset \omega_1$ or $\omega_1 \sqsubseteq \mathbb{R}$. The former cannot hold, because $\mathsf{WO}(\mathbb{R})$ implies that $\omega_1 \preceq \mathbb{R}$ So $\omega_1 \cong A \subseteq \mathbb{R}$, and since A must be uncountable, wCH gives $A \sim \mathbb{R}$, hence $\omega_1 \sim \mathbb{R}$. ☐

As mentioned, ω_1 by definition has an ordering that is initially countable. But it is also a well-order. We would like to distinguish between well-founded an not well-founded initially countable orderings on $\mathbb{R}$ and therefore consider the following two statements:

ICO There exists an initially countable order on $\mathbb{R}$.

ICWO There exists an initially countable *wellorder* on $\mathbb{R}$.

Clearly, ICWO implies ICO (and they are equivalent when $\mathbb{R}$ is well-orderable). This is no surprise, since ICWO is a rather strong statement: it implies $\mathsf{WO}(\mathbb{R})$ and as a result, it is even equivalent to CH:

THEOREM 4.5. $\mathsf{ZF} \vdash (\mathsf{ICWO} \longleftrightarrow \mathsf{CH})$.

Proof. For the right to left direction, transfer the order on ω_1 to $\mathbb{R}$ using the bijection given by CH to prove that ICWO holds. Now suppose ICWO holds and that $<$ is an initially countable well-order on $\mathbb{R}$. Again this implies $\omega \preceq \mathbb{R}$. Suppose this is 'strict', i.e. that $\omega_1 \cong \mathrm{IS}(x)$ for some $x \in \mathbb{R}$. By assumption $\mathrm{IS}(x)$ is countable, so injects into ω. But then ω_1 injects into ω, contradicting it's definition. We conclude that $\mathbb{R} \cong \omega_1$, which implies CH. ☐

It is unknown to the author what the preciese relation between ICO and wCH is; that might be an interesting question.

4.3. Geometric Sierpiński partition

Let us now return to the Sierpiński partition. Not only does the partition behave strangely from a measure-theoretic perspective, it also has peculiar geometric properties.[2] Call a

[2] cf. Kharazishvili 2004; Komjáth and Totik 2006.

partition $\{A, B\}$ or $\mathbb{R}^2$ in which A has countable intersection with every horizontal line and B with every vertical line a *geometric Sierpiński partition*. We look at its relation to the Sierpiński partition, non-measurable sets and CH.

PROPOSITION 4.6. *If $\{A, B\}$ is a Sierpiński partition of $\mathbb{R}^2$, then $\{A, B\}$ is a geometric Sierpiśki partition.*

Proof. Denote the initially countable ordering on $\mathbb{R}$ by $\prec$. Let $L = \{(x, c) : x \in \mathbb{R}\}$ be a horizontal line for fixed $c \in \mathbb{R}$. For all $x \in \mathbb{R}$ we have $(x, c) \in A$ if and only if $x \prec c$, and there are only countably many, hence $L \cap A$ is countable. The proof for vertical lines is similar. □

PROPOSITION 4.7. *If $\{A, B\}$ is a geometric Sierpiński partition, A and B are non-measurable.*

Proof. The proof is very similar to the proof of non-measurability of a Sierpiński partition. Since $A \cap \mathbb{R}^2 \times \{y\} = A^y \times \{y\}$ is countable, A^y must be countable, for every $y \in \mathbb{R}$. So the set of y for which $\mu(A^y) > 0$ is a nullset. Similarly for B and Theorem 4.1 shows that $\mu(\mathbb{R}^2) = 0$, which is absurd. □

THEOREM 4.8. *Assume $\mathsf{UC}_{\omega_1}(\omega_1 \times \mathbb{R})$ and $\omega_1 \preceq \mathbb{R}$. If $\mathbb{R}^2$ has a geometric Sierpiński partition $\{A, B\}$, then the Continuum Hypothesis holds.*

Proof of Theorem 4.8. Let $f : \omega_1 \to X \subseteq \mathbb{R}$ be the bijection given by $\omega_1 \subseteq \mathbb{R}$, where $X = \mathrm{ran}(f)$. Now define the set $Z = (X \times \mathbb{R}) \cap B$.

$$Z = \bigcup_{x \in X} (\{x\} \times \mathbb{R}) \cap B = \bigcup_{x \in X} \{x\} \times B_x \sim \bigcup_{y \in \omega_1} \{y\} \times B_{f^{-1}(y)}$$

Note that $\{x\} \times \mathbb{R}$ is a vertical line that has countable intersection with B, so $\{x\} \times B_x$ is countable for each x. Now $\mathsf{UC}_{\omega_1}(\omega_1 \times \mathbb{R})$ implies that $Z \preceq \omega_1$. It is moreover clear that $X \preceq Z$, so it follows that $Z \sim \omega_1$.

Next we claim that the projection $\mathrm{Pr}_2(Z)$ covers whole $\mathbb{R}$, implying that $\mathbb{R}$ injects into ω_1. This is illustrated in Figure 4.2. Let $y \in \mathbb{R}$; we proceed by showing that $y \in \mathrm{Pr}_2(Z)$. On the one hand, $(\mathbb{R} \times \{y\}) \cap A = A^y \times \{y\}$ must be countable since it is the intersection of a horizontal line with A. On the other hand $(\mathbb{R} \times \{y\}) \cap (X \times \{y\}) = X \times \{y\}$ is in bijection to ω_1, which is uncountable. So there must be points of $X \times \{y\}$ outside A but inside $X \times \{y\}$. Such a point (t, y) must be in B (it is not in A) and an element of $X \times \mathbb{R}$. So $(t, y) \in Z$ and it follows that $y \in \mathrm{Pr}_2(Z)$. Thus $\mathbb{R} \subseteq \mathrm{Pr}_2(Z)$ and it follows that $\mathbb{R} \sim \omega_1$. □

Conclusion To summarise and simplify the last result: in ZFC the existence of a geometric Sierpiński partition is equivalent to the Continuum Hypothesis. The more subtle relations we found in this chapter are illustrated in Figure 4.3. We have found a relation between the Continuum Hypothesis and the Sierpiński Partition, which turned out to be a non-measurable set. The non-measurability relied, just as with Bernstein sets, on countability of certain sets. This was ensured by an initially countable ordering on $\mathbb{R}$, rather than by topological or algebraic properties of the measure.

In the next, final chapter, we finish our investigations to non-measurability with a purely set-theoretic exposition.

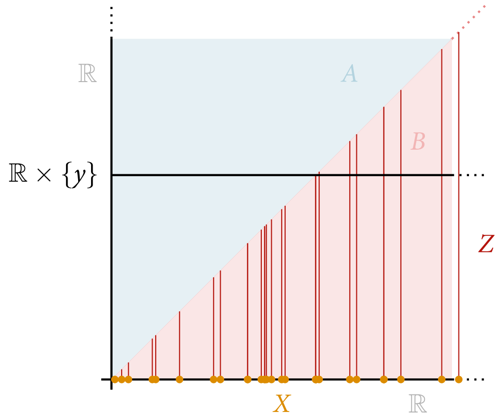

Figure 4.2.: Here $\mathbb{R}^2$ is partitioned in sets A and B. X is a subset of $\mathbb{R}$ of cardinality ω_1 and $Z = (X \times \mathbb{R}) \cap B$ corresponds to the red lines. The black line $\mathbb{R} \times \{y\}$ has countably many points in A, but uncountably many in $X \times \{y\}$, so it contains points in B.

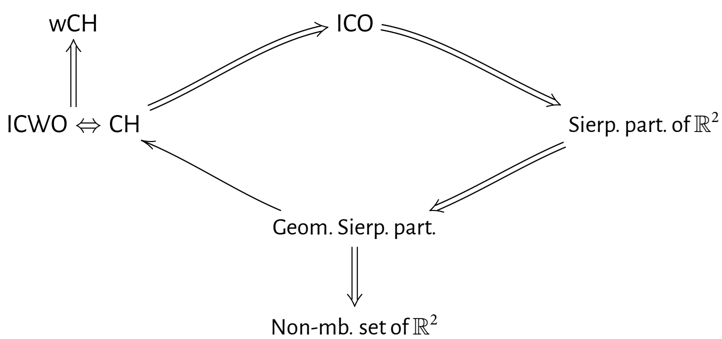

Figure 4.3.: Summary of the relations found in this chapter. The double-lined arrows are implications provable in ZF. The single arrow makes use of the additional assumptions of $\mathrm{UC}_{\omega_1}(\omega_1 \times \mathbb{R})$ and $\omega_1 \preceq \mathbb{R}$. An interesting questions might ask for the relation between wCH and ICO.

CHAPTER 5

Real measurable cardinals

So far, we have seen several constructions of non-measurable subsets of $\mathbb{R}$, relying on first algebraic, topological and finally ordering properties of the reals. In this final chapter we want to present a completely set-theoretical reason for the existence of non-measurable sets. Eventually, have to relate ordinals and the reals, and therefore the final results in this chapter will assume the Continuum Hypothesis.

First we consider real-valued measurable cardinals. Using Ulam's matrix, we can show that these are weakly inaccessible, which will show that under CH there cannot exist weak measure defined on $\mathcal{P}(\mathbb{R})$.

5.1. Measurable cardinals

In chapter 1 we asked for which sets X there exists a total weak measure, a question that was at that point postponed because deep set theoretic considerations were involved. At that point we continue in this chapter.

The reason that we are naturally led to set theory when posing the question is the following. If X and Y are two sets, then any bijection $f : X \to Y$ (if one exists) can transfer a weak measure μ_X on $\mathcal{P}(X)$ to a weak measure on $\mathcal{P}(Y)$ by defining

$$\mu_Y(A) := \mu_X(f^{-1}[A]), \qquad A \subseteq Y$$

as is easily verified. So in ZFC there exists a weak total measure on X if and only if there exists one on the cardinal $\mathrm{Card}(X)$. The question can thus be rephrased as: for which cardinals κ does there exist a weak measure defined on $\mathcal{P}(\kappa)$? If we can answer this question, it will give some purely set theoretic insight into measures on $\mathbb{R}$, but only if we know how to relate $\mathbb{R}$ to cardinals.

Clearly, κ cannot be finite or countable by countable additivity of a measure. Let us immediately fully generalise that notion, as anticipated in chapter 1

DEFINITION 5.1. Let κ be an uncountable cardinal. A weak measure on a set X is called *kappa-additive*[1] if $\mu(\bigcup \mathcal{N}) = 0$ for every family $\mathcal{N}$ of subsets of X such that $|\mathcal{N}| < \kappa$ and $\mu(N) = 0$ for ever $N \in \mathcal{N}$.

Note that a countable additive measure is *not* $\aleph_0$-additive, but $\aleph_1$-additive. A finitely additive measure is $\aleph_0$-additive.

DEFINITION 5.2. We call a uncountable cardinal κ *real-valued measurable* if there exists a weak, κ-additive measure on $\mathcal{P}(\kappa)$.

Such cardinals must be regular. By definition, an infinite cardinal κ is *regular* if it equals its *cofinality*, i.e. $\mathrm{cf}(\kappa) = \kappa$. Equivalently, κ is regular if there exists no set $\{\theta_\alpha : \alpha < \gamma\}$ with $\theta_\alpha < \kappa$ and $\gamma < \kappa$ such that $\kappa = \bigcup_{\alpha<\gamma} \theta_\alpha$. Now suppose towards a contradiction that κ is real-valued measurable, but not regular. Then there exists some $\gamma < \kappa$ and a set $\{\theta_\alpha \in \kappa : \alpha < \gamma\}$ with $\kappa = \bigcup_{\alpha<\kappa} \theta_\alpha$. By κ-additivity and diffuseness we see that $\theta_\alpha = \{\nu : \nu < \theta_\alpha\}$ is a set of measure zero for every $\alpha < \gamma$. Now it follows from the definition of κ-additivity that $\mu\big(\bigcup_{\alpha<\kappa} \theta_\alpha\big) = \mu(\kappa) = 0$, which is absurd. So we have shown:

THEOREM 5.3. *If κ is real-valued measurable, then κ is a regular cardinal.*

More is true. A real-valued measurable cardinal will also be a limit cardinal. There one obvious example of a regular limit cardinal, $\aleph_0$, is countable so does not qualify as a real valued measurable cardinal. So do regular limit cardinals exist at all? In fact, it is not provable in ZFC that uncountable regular limit cardinals exist[2] and for that reason such cardinals are called *weakly inaccessible cardinals*. We devote the rest of this section to

[1] cf. Just and Weese 1995.
[2] Devlin 2000, p. 95.

$$\kappa\text{ rows}\left\{\begin{bmatrix} A_0^0 & A_1^0 & A_2^0 & \cdots & A_\alpha^0 & \cdots \\ A_0^1 & A_1^1 & A_2^1 & \cdots & A_\alpha^1 & \cdots \\ A_0^2 & A_1^2 & A_2^2 & \cdots & A_\alpha^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ A_0^\nu & A_1^\nu & A_2^\nu & \cdots & A_\alpha^\nu & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}\right. = \underbrace{\begin{bmatrix} \{\xi : f_\xi(0)=0\} & \{\xi : f_\xi(1)=0\} & \cdots & \{\xi : f_\xi(\alpha)=0\} & \cdots \\ \{\xi : f_\xi(0)=1\} & \{\xi : f_\xi(1)=1\} & \cdots & \{\xi : f_\xi(\alpha)=1\} & \cdots \\ \{\xi : f_\xi(0)=2\} & \{\xi : f_\xi(1)=2\} & \cdots & \{\xi : f_\xi(\alpha)=2\} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \\ \{\xi : f_\xi(0)=\nu\} & \{\xi : f_\xi(1)=\nu\} & \cdots & \{\xi : f_\xi(\alpha)=\nu\} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{\kappa^+\text{ columns}}$$

Figure 5.1.: For κ and uncountable cardinal with successor κ^+, the $\kappa \times \kappa^+$ Ulam matrix $\mathcal{U}_\kappa = \{A_\alpha^\nu : \alpha < \kappa^+ \text{ and } \nu < \kappa\}$ is shown. It has κ rows and κ^+ columns. The sets $A_\alpha^\nu = \{\xi : f_\xi(\alpha) = \nu\}$ are defined using bijections f_ξ from ξ to a subset of κ. As a result, any two sets in a given row are disjoint and the union of all sets in a given column equals κ^+, except for a countable set. (See the proof of lemma 5.6 for details.)

proving that real-valued measurable cardinals are weakly inaccessible. For that we need a so called *Ulam matrix*, but first: a peculiar lemma.

LEMMA 5.4. *Let μ be a finite measure (not necessarily diffuse) on X. Then any collection of disjoint sets with positive measure, is at most countable.*

Proof. Let $\mathcal{A} = \{A_\alpha : \alpha < \gamma\}$ be an uncountable collection of disjoint sets of positive measure. Split this collection up into different subcollections by

$$\mathcal{P}_n := \{A_\alpha \in \mathcal{A} : \; \mu(A_\alpha) > \frac{1}{n}\}.$$

Some $\mathcal{P}_N$ must contain uncountably many sets, otherwise $\mathcal{A}$ could be a countable union of countable sets, hence countable. Let $C \subsetneq \mathcal{P}_N$ be a countable subset of $\mathcal{P}_N$. The A_α's in C are pairwise disjoint and all have measure greater than $1/N$. By countable additivity, $\mu(\bigsqcup C) = \infty$, which is impossible since X has finite measure. □

5.2. Ulam's matrix

DEFINITION 5.5. Let κ be an infinite cardinal, and κ^+ its successor. The set

$$\mathcal{U}_\kappa := \{A_\alpha^\nu : \; \alpha < \kappa^+ \text{ and } \nu < \kappa\} \tag{5.1}$$

of subsets of κ^+ is called an *Ulam matrix on κ^+* if it satisfies

(U1) Each 'row' consists of pairwise disjoint sets. That is: for any fixed $\nu < \kappa$ we have $A_\alpha^\nu \cap A_\beta^\nu = \emptyset$ for all $\alpha \neq \beta$ smaller than κ^+.

(U2) Each 'column' consists of pairwise disjoint sets. That is: for fixed $\alpha < \kappa^+$ we have $A^\nu_\alpha \cap A^\eta_\alpha = \emptyset$ for $\nu \neq \eta$ smaller than κ.

(U3) The union of each 'column' differs from κ^+ by a set of at most cardinality κ. That is: for any fixed $\alpha < \kappa^+$ we have $\bigcup\{A^\nu_\alpha : \nu < \kappa\} = \kappa^+ \setminus S_\alpha$ where S_α is a set of at most cardinality κ.

The name suggests that we should think of $\mathcal{U}_\kappa$ as an infinite *matrix*. This is illustrated in Figure 5.1. There the following result is already used, namely, that such a matrix exists at all.

LEMMA 5.6 (ULAM). *If κ is an infinite cardinal, there exists an Ulam matrix on κ^+.*

Proof. Note that if $\xi < \kappa^+$, then the cardinality of ξ will be at most κ. As a result, we can find a bijection f_ξ from ξ to a subset of κ — or, to be precise, to an element smaller than κ. Fix a function f_ξ for every $\xi < \kappa$. This is indeed a very explicit use of the axiom of choice, but we are working in ZFC anyway. With these functions we define for each $\xi < \kappa^+$ and $\nu < \kappa$ the following sets

$$A^\nu_\alpha := \{\xi \in \kappa^+ : f_\xi(\alpha) = \nu\}.$$

Note that $f_\xi(\alpha)$ is only defined if $\alpha < \xi$. Whenever $\xi \in A^\nu_\alpha$ it thus follows that $\alpha < \xi$. We show that the set of all A^ν_α is an Ulam matrix on κ^+.

(U1) Suppose that ξ is in both A^ν_α and A^ν_β, for $\alpha, \beta < \xi$ and $\nu < \kappa$. Then $f_\xi(\alpha) = \nu$ and $f_\xi(\beta) = \nu$. Since f_ξ is injective, it follows that $\alpha = \beta$.

(U2) Suppose that ξ is in both A^ν_α and A^η_α for $\nu, \eta < \kappa$ and $\alpha < \kappa^+$. Then $\eta = f_\xi(\alpha) = \nu$ and it follows that $\nu = \eta$.

(U3) An element ξ is in $\bigcup\{A^\nu_\alpha : \nu < \kappa\}$ if and only if it is in some A^ν_α for $\alpha < \xi$ and $\nu < \kappa$. But this is the case exactly when $f_\xi(\alpha) = \nu$ for some $\nu < \kappa$. Clearly, this is only possible if $\xi > \alpha$, or equivalently, if

$$\xi \in \kappa \setminus \{\zeta : \zeta \leq \alpha\} = \kappa \setminus \alpha + 1.$$

Since $\alpha + 1 < \kappa^+$, its cardinality is at most κ. So we have proved (U2) for $S_\alpha = \alpha + 1$.

We have shown that $\mathcal{U}_\kappa = \{A^\nu_\alpha\}$ is an Ulam matrix. □

Now that we have this device at our disposal, we can prove that real-valued measurable cardinals are weakly inaccessible:

THEOREM 5.7 (ULAM, 1930). *If κ is a real-valued measurable cardinal, then κ is weakly inaccessible.*

Proof. Let κ be a cardinal and μ a weak measure on $\mathcal{P}(\kappa)$. By Theorem 5.3 κ must be regular, so we only need to show that κ is a limit. Arguing towards a contradiction, suppose that κ is not a limit cardinal, but a successor instead, say $\kappa = \lambda^+$. Then we can define an Ulam matrix $\mathcal{U}_\lambda$, such as defined in (5.1). Let's fix the notation $A_\alpha := \{A^\nu_\alpha : \nu < \lambda\}$ for the α-th column and $A^\nu := \{A^\nu_\alpha : \alpha < \lambda^+\}$ for the ν-th row. We make two claims.

CLAIM A. *Every column contains a set of positive measure: for every $\alpha < \lambda^+$ there exists some ν_α such that $\mu(A_\alpha^{\nu_\alpha}) > 0$.*

Proof of Claim. Suppose not; then $\mu(A_\alpha^\nu) = 0$ for all $\nu < \lambda$. Property (U2) ensures that the sets in the column A_α are moreover pairwise disjoint so it follows from κ-additivity that $\bigsqcup A_\alpha$ has measure zero. However, property (U3) gives that the complement set has at most cardinality λ. Such a set has measure zero with respect to the λ^+-additive measure μ, so we have found that

$$\mu(\lambda^+) = \mu\Big(\Big(\bigsqcup A_\alpha\Big) \sqcup S_\alpha\Big) = \mu\Big(\bigsqcup A_\alpha\Big) + \mu(S_\alpha) = 0,$$

where $S_\alpha = \lambda^+ \setminus \bigsqcup A_\alpha$ denotes the complement set. This contradicts the assumption that $\mu(\lambda^+) = 1$. ◁

CLAIM B. *There is a row containing λ^+ sets of positive measure. That is, there is a fixed $\eta < \lambda$ such that $|\{\alpha : \mu(A_\alpha^\eta) > 0\}| = \lambda^+$.*

Proof of Claim. If not, every row A^ν contains at most λ-many sets of positive measure. Since there are precisely λ rows — one for each $\nu < \lambda$ — there can be no more than λ sets of positive measure in the whole matrix $\mathcal{U}$. However, the matrix contains λ^+ of these sets, since by CLAIM A each of the λ^+-many columns contains one. ◁

We are done: by (U1) the sets in each row are pairwise disjoint, so CLAIM B contradicts LEMMA 5.4, which states that there are at most countably many disjoint sets with positive measure. □

To repeat the result, all real-valued measurable cardinals are apparently weakly inaccessible. How does this relate to measures on $\mathbb{R}$? Suppose that we have a weak measure defined on $\mathcal{P}([0,1])$. Since $[0,1]$ is in bijection with $2^{\aleph_0}$, we can transfer the measure to $\mathcal{P}(2^{\aleph_0})$ and name it μ. Clearly, μ is $\aleph_1$-additive as this is true for any measure. If, on the other hand, we suppose that μ is κ-additive for $\kappa > 2^{\aleph_0}$, we reach the contradiction

$$\mu(2^{\aleph_0}) = \mu\Big(\bigsqcup_{\alpha < 2^{\aleph_0}} \{\alpha\}\Big) = 0,$$

by **??** (and indeed, diffuseness of μ is indispensable). There must thus be a greatest uncountable cardinal κ such that μ is κ-additive, and $\aleph_1 \leq \kappa \leq 2^{\aleph_0}$.

Fix that κ. Its definition ensures that we can find a collection of disjoint measure zero sets $\{A_\nu : \nu < \kappa\}$ such that their union A has positive measure. Then define the map $f\colon A \to \kappa$ by $f(a) = \nu$ if $a \in A_\nu$ and the measure

$$\mu_\kappa(B) := \frac{\mu(f^{-1}[B])}{\mu(A)}, \qquad B \subseteq \kappa.$$

Indeed, μ_κ is nonzero, but also diffuse since $\mu_\kappa(\{\nu\}) = \mu(A_\nu) = 0$. If $\{E_\nu : \nu < \gamma\}$ for $\gamma < \kappa$ is a collection of disjoint sets, it is easily verified that

$$\mu\Big(f^{-1}\Big[\bigsqcup_{\nu<\gamma} E_\nu\Big]\Big) = \mu\Big(\bigsqcup_{\nu<\gamma} f^{-1}[E_\nu]\Big) = \sum_{\nu<\gamma} \mu(f^{-1}[E_\nu]),$$

from which the κ-additivity of μ_κ follows directly. In other words, κ is a real-valued measurable cardinal and by Theorem 5.7 weakly inaccessible. Summarising, we have shown that the existence of a total measure on $\mathcal{P}([0,1])$ implies that there exist a weakly inaccessible cardinal $\kappa \leq 2^{\aleph_0}$.

Now suppose that the Continuum Hypothesis holds, i.e. $\aleph_1 = 2^{\aleph_0}$. Then $\aleph_1 \leq \kappa \leq 2^{\aleph_0} = \aleph_1$ in particular implies that κ is a successor cardinal, which is false. So the Continuum Hypothesis must fail. All this is the content of the following theorem.

THEOREM 5.8. *Assume there exists a weak measure defined on $\mathcal{P}([0,1])$. Then there exists a weakly inaccessible cardinal $\aleph_1 \leq \kappa \leq 2^{\aleph_0}$. Consequently, if* CH *holds, there cannot exist a weak measure defined on $\mathcal{P}([0,1])$.*

Note that instead of assuming CH, it suffices to assume that there exists no weakly inaccessible cardinal below $2^{\aleph_0}$.

Conclusion Ulam answered Banach's Generalised Measure Problem: the sets that allow a total weak measure are the real-valued measurable cardinals and if they exist, they must be weakly-inaccessible. This showed that if the Continuum Hypothesis holds, then there cannot be a total measure on $\mathcal{P}(\mathbb{R})$. In other words, we will not be able to define a total measure on $\mathbb{R}$, even if it does not have to extend the Lebesgue measure. Therefore, the earlier constructions of non-measurable sets were necessary: there always will be non-measurable sets, if we at least assume CH.

CHAPTER 6

Concluding remarks

When and how can we construct sets that have no length, no measure? With that question, we started this thesis. The first part, the when, was already answered in the very first chapter: only for weak countably additive measures can we hope find non-measurable sets, if we work in ZF plus some form of choice. In the chapters that followed we examined three well-known constructions of non-measurable sets, due to Vitali, Bernstein and Sierpiński. Finally the result by Ulam showed that assuming CH, no weak measure can be totally defined on $[0, 1]$. All these results used different fragments of choice, and a summary is provided in Figure 6.1.

Roughly speaking, Vitali's construction crucially used algebraic properties of the measure, Bernstein's construction topological and finally Sierpiński used the structure of $\mathbb{R}$. Dense subgroups, perfect sets and initially countable orderings, those were the ingredients that played an important rôle. Hopefully this thesis highlighted some of these concepts and showed how they are related to the existence of non-measurable sets.

Only when moving on to pure set theory and the theory of measurable cardinals, we left specific considerations of $\mathbb{R}$ behind us. And as mentioned before, what we have seen of measurable cardinals is only a very limited portion of the existing theory. But that is good news: there is still much more left to explore. And this is true for all of the topics we covered. Many more interesting results lie ahead.

For example, I orginially intended to include constructions of non-measurable sets from a non-principal ultrafilter over ω and a construction using only an injection $\omega \to \mathbb{R}$, as suggested by my supervisor. Due to limitations, both in scope and time, I have not been able to examine these in more detail, but these topics could form a good continuation of this text.

Furthermore, at least two set-theoretic questions remained unanswered. The first, encountered in the chapter on Vitali sets, asked if a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$ implies the existence of a well-ordering of $\mathbb{R}$ in ZF. The second question asked for the relation between the existence of initially countable orderings of $\mathbb{R}$ and the weak Continuum Hypothesis. It might be of interest to further look into these, and perhaps using more advanced theory than was available to us here could provide an answer.

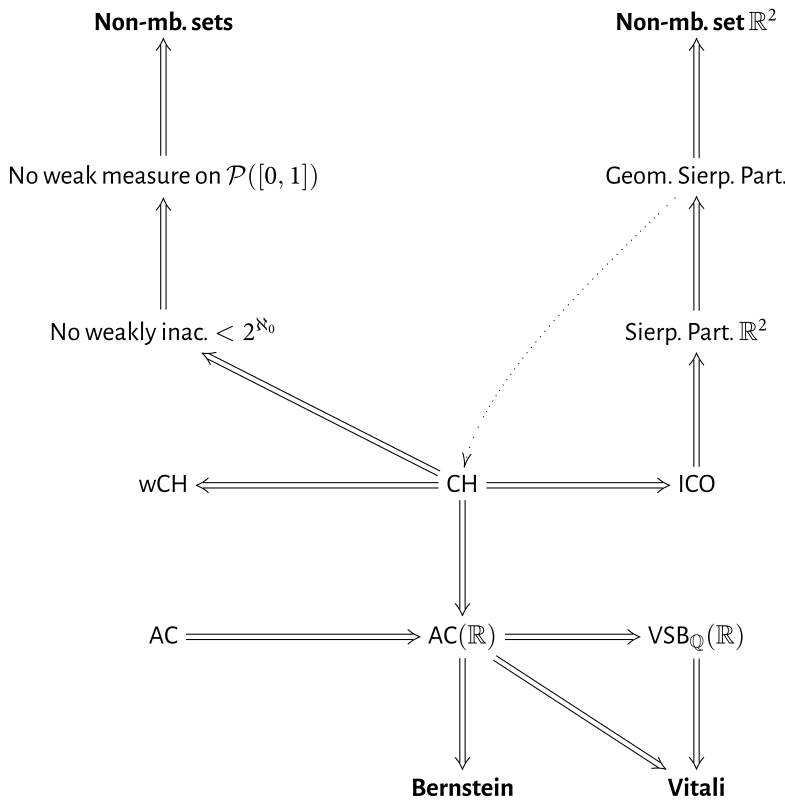

Figure 6.1.: Relations found between axioms and non-measurable sets. All double-lined implications are provable in ZF, again accept for the single line which assumes $\mathsf{UC}_{\omega_1}(\omega_1 \times \mathbb{R})$ and $\omega_1 \preceq \mathbb{R}$.

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Appendices

CHAPTER A

Omitted Proofs

Well-orders

THEOREM A.1. *If (X, R) and (Y, S) are well-orders, I and I' initial segments of Y and $f : X \to I$ and $f' : X \to I'$ are isomorphisms, then $I = I'$ and $f = f'$.*

Proof. By induction; consider $A = \{x \in X : f(x) = f'(x)\}$. Let $x \in X$ and suppose that $\mathrm{IS}(x) \subseteq A$, which holds if and only if $f(z) = f(z')$ for all $z \neq x$ such that zRx. Suppose towards a contradiction that $f(x) \neq f'(x)$. As R is total we may without loss of generality assume that $f(x)Rf'(x)$. It follows that $f(x) \in \mathrm{ran}(f') = I'$, because I' is an initial segment. But then $f(x) = f'(x')$ for some $x' \in X$. By the induction hypothesis, $f(z) = f'(z)$ for all z such that zRx, so we must have xRx'. This means

$$f'(x') = f(x)Rf'(x) \quad\implies\quad f'(x')Rf'(x)$$

which contradicts the fact that f' is an isomorphism: the order is changed! □

COROLLARY A.2. *No well-order is isomorphic to a proper initial segment.*

Proof. Let (X, R) be a well-order. Clearly X is a (nonproper) initial segment of X and (X, R) is isomorphic to it. But then it cannot be isomorphic to a proper initial segment $\mathrm{IS}(x)$ as that would contradict the uniqueness proven in the previous theorem. □

THEOREM A.3 (FUNDAMENTAL THEOREM OF WELL-ORDERS). *For any two well-orders (X, R) and (Y, S) exactly one of the following three holds: (1) $X \sqsubset Y$; (2) $X \cong Y$; or (3) $Y \sqsubset X$.*

BIBLIOGRAPHIC NOTE. Based on the lectures by **lowe**

Proof. Note that Theorem A.1 indicates that exactly one of $X \sqsubset Y$, $X \cong Y$ and $Y \sqsubset X$ holds. Now recursively define a function

$$f(x) = \begin{cases} \min_Y\big(Y \setminus \mathrm{ran}(f \restriction \mathrm{IS}(x))\big) & \text{if } Y \setminus \mathrm{ran}(f \restriction \mathrm{IS}(x) \text{ is nonempty} \\ \textsc{stop} & \text{otherwise,} \end{cases}$$

where $\textsc{stop} \notin Y$. So $f : X \to Y \cup \{\textsc{stop}\}$. Let $I = \{x : f(x) \in Y\}$. Clearly, I is an initial segment of X. Moreover, $J := \mathrm{ran}(f) \cap Y$ must be an initial segment of Y. To see this, suppose that $y_0 S y_1$ and $y_1 \in J$. Then there must be an x such that $f(x) = y_1$. If y_0 would not be in J, then $y_0 \notin \mathrm{ran}(f)$ and in particular $y_0 \notin \mathrm{ran}(f \restriction \mathrm{IS}(x))$, contradicting the definition of f.

Next we claim that either I or J is a proper initial segment. Suppose we have $I = \mathrm{IS}(x)$ with $x \notin I$. Then $f(x) \notin Y$, which means that $f(x) = \textsc{stop}$, hence $Y = \mathrm{ran}(f \restriction \mathrm{IS}(x))$. But now we find the following inclusions

$$J = \mathrm{ran}(f) \cap Y \supseteq \mathrm{ran}(f \restriction \mathrm{IS}(x)) \cap Y = Y \cap Y = Y,$$

so J is not a proper initial segment.

To finish the proof, note that if I is proper, then $J = Y$ and $f \restriction I$ shows that $Y \sqsubset X$. If J is proper, then $I = X$ and f shows $X \sqsubset Y$. Otherwise, if $I = X$ and $J = Y$, f is an isomorphism so $X \cong Y$. □

Axiom of choice

THEOREM A.4. $\mathsf{ZF} \vdash (\mathsf{AC}(X) \longleftrightarrow \mathsf{WO}(X))$.

Proof. For the right to left direction, let $\mathcal{F}$ be a set of nonempty subsets of X; By assumption there exists a well-order $<$ on X. Every $F \in \mathcal{F}$ is a nonempty subset of X and thus has a (unique) least element with respect to $<$. So we can define a choice function $c(F) = \min F$ on $\mathcal{F}$, hence $\mathsf{AC}(X)$ holds.

Conversely, let c by $\mathsf{AC}(X)$ be a choice function of $\mathcal{P}(X) \setminus \{\emptyset\}$. Using transfinite recursion, let $\pi(\alpha)$ be $c(X \setminus \mathrm{ran}(\pi \restriction \alpha))$ if this set is nonempty and $\textsc{stop} \notin X$ otherwise. If $\alpha < \beta$ are ordinals not mapped to $\textsc{stop}$ then $\pi(\alpha) \neq \pi(\beta)$. To see this, note that $\alpha \in \beta = \mathrm{dom}(\pi \restriction \beta)$. So $\pi(\alpha) \in \mathrm{ran}(\pi \restriction \beta)$. But $\pi(\beta) \notin \mathrm{ran}(\pi \restriction \beta)$ by definition, so they must be different. Moreover, there must exist an ordinal such that $\pi(\alpha) = \textsc{stop}$. Otherwise, $X \setminus \mathrm{ran}(\pi \restriction \alpha)$ would be nonempty for all α, and since π is injective (outside $\textsc{stop}$) that implies that X must be a proper class. So let α be the least ordinal such that $\pi(\alpha) = \textsc{stop}$. Then $\pi \restriction \alpha : \alpha \to X$ is a bijection, so we can well-order X. □

Lebesgue measure determined on unit interval

THEOREM A.5. *There exists a total, translation invariant measure on* $\mathbb{R}$ *if and only if there exists a measure defined on* $\mathcal{P}([0,1])$ *that is translation invariant on* $[0,1]$*: if* A *and* $A + x$ *are subsets of* $[0,1]$*, then their measures coincide.*

Proof. One direction is immediate: a measure on $\mathbb{R}$ can be restricted to $[0, 1]$. Conversely, suppose that $\mu : \mathcal{P}([0, 1)) \to [0, +\infty]$ satisfies the conditions. Define the measure of a set $A \subseteq \mathbb{R}$ by measuring its parts in each interval $[n, n+1)$. Formally, first fix an enumeration $\{z_n : n \in \mathbb{N}\}$ of $\mathbb{Z}$ and then define

$$\mu_{\mathbb{R}}(A) := \sum_{n\in\mathbb{N}} \mu\Big(([z_n, z_n + 1) \cap A) - z_n\Big),$$

Note that this is indeed defined for all subsets of $\mathbb{R}$, since μ is a total measure on $[0, 1)$. It is moreover countably additive because disjoint sets $A_1, A_2, \ldots$ have total measure

$$\begin{aligned}
\mu_{\mathbb{R}}\Big(\bigsqcup_{i\in\mathbb{N}} A_i\Big) &= \sum_{n\in\mathbb{N}} \mu\Big(([z_n, z_n + 1) \cap \bigsqcup_{i\in\mathbb{N}} A_i) - z_n\Big) \\
&= \sum_{n\in\mathbb{N}} \mu\Big(\bigsqcup_{i\in\mathbb{N}} ([z_n, z_n + 1) \cap A_i - z_n)\Big) \\
&= \sum_{n\in\mathbb{N}} \sum_{i\in\mathbb{N}} \mu\Big([z_n, z_n + 1) \cap A_i - z_n\Big) = \sum_{i\in\mathbb{N}} \mu_{\mathbb{R}}(A_i).
\end{aligned}$$

Translation invariance is not difficult but we have to deal with some technicalities. First observe that for set X that is a subset of some interval $[z_n, z_n+1]$ we have $\mu_{\mathbb{R}}(X) = \mu(x-z_n)$. Now let $0 \leq \alpha < 1$. and define

$$X_1 := X + \alpha \cap [z_n, z_n + 1) \qquad \text{and} \quad X_2 := X + \alpha \cap [z_n + 1, z_n + 2).$$

Such that $X + \alpha = X_1 \sqcup X_2$. Moreover note that $X_1 - x$ is a subset of $[z_n, z_n + 1)$ and $X_2 + (1 - x)$ is a subset of $[z_n + 1, z_n + 2)$. Now we calculate

$$\mu_{\mathbb{R}}(X_1) = \mu(X_1 - z_n) = \mu(X_1 - \alpha - z_n)$$

and by comparable arguments, $\mu_{\mathbb{R}}(X_2)$ equals

$$\mu(X_2 - (z_n + 1)) = \mu(X_2 + (1 - \alpha) - z_n - 1) = \mu(X_2 - \alpha - z_n).$$

By additivity we have $\mu_{\mathbb{R}}(X + \alpha) = \mu_{\mathbb{R}}(X_1) + \mu_{\mathbb{R}}(X_2)$ and this equals

$$\mu(X_1 - \alpha - z_n) + \mu(X_2 - \alpha - z_n) = \mu((X_1 \sqcup X_2) - \alpha - z_n) = \mu_R(X).$$

To summarize we have seen that for any set X in some interval $[z_n, z_n + 1$ we have $\mu_{\mathbb{R}}(X + \alpha) = \mu_{\mathbb{R}}(X)$ for $0 \leq \alpha < 1$. Analogue reasoning shows this is also true for $-1 < \alpha \leq 0$. Now let x be any real and write $x = \alpha_1 + \cdots + \alpha_n$ where α_i are all positive or all negative reals with at most absolute 1. Then

$$\begin{aligned}
\mu(X + \alpha) &= \mu((A + \alpha_1 + \cdots + \alpha_{n-2} + \alpha_{n-1}) + \alpha_n) \\
&= \mu((A + \alpha_1 + \cdots + \alpha_{n-2}) + \alpha_{n-1}) = \cdots = \mu(A),
\end{aligned}$$

and the proof is complete. □

Saturated non-measurable set based on Vitali

Recall that a set $E \subseteq \mathbb{R}$ is called *saturated non-measurable* if for every Lebesgue measurable subset A of $\mathbb{R}$ the following holds:

$$m_*(E \cap A) = 0 \quad \text{and} \quad m^*(E \cap A) = m(A).$$

We have the following characterization[1]

THEOREM A.6. *A set $E \subseteq \mathbb{R}$ is saturated non-measurable if and only if any of the following conditions hold:*

(i) $m_*(E) = 0 = m_*(E^c)$.

(ii) *$A \cap E$ is non-measurable for every measurable set $A \subseteq \mathbb{R}$ of positive measure.*

Proof.

(def) $\implies$ (i) Clearly, for all E saturated nonmeasurable, $m_*(E \cap \mathbb{R}) = m_*(E) = 0$. For $m_*(E^c) = 0$, let F be a compact subset of E^c. Then by definition of E, $m^*(F) = m^*(F \cap E)$. This is zero, since F and E are disjoint. It follows that $m_*(E^c) = 0$.

(i) $\implies$ (def) Let $A \subseteq \mathbb{R}$ be a measurable set and F a compact set inside $A \cap E \subseteq E$. Then $m^*(F) = 0$ because all compact sets in E have measure zero. As a result $m_*(A \cap E) = \sup\{m^*(F) : F \subseteq A \cap E \text{ compact}\} = 0$.

Next, let G be an open set containing $A \cap E$. All compact subsets of E^c have measure zero by assumption. In particular $m^*(G^c \cap [-n, n]) = 0$ for all $n \in \mathbb{N}$, so G^c has zero inner measure. Because G^c is measurable its inner- and outer measure are equal and zero. Caráthéodory's condition gives

$$m^*(E) = m^*(E \cap G) + m^*(E \cap G^c) = m^*(E \cap G),$$

as desired.

(def) $\implies$ (i) Clearly, by (i), $m_*(A \cap E) \le m_*(E) = 0$, so we only have to show that $m^*(A \cap E) > 0$. But that must be the case, since E is saturated non-measurable, so $m^*(A \cap E) = m^*(A) > 0$.

(ii) $\implies$ (def) Let $E \subseteq \mathbb{R}$ such that for all $A \subseteq \mathbb{R}$, $A \cap E$ is non-measurable. Suppose that $m_*(E) > 0$. That means that there exists a closed set $F \subseteq E$ with $m(F) > 0$. But by definition of E, $E \cap F = F$ must be non-measurable, which is impossible since all closed sets are measurable. So $m_*(E) = 0$. Next suppose that $m_*(E^c) > 0$. Then there exists a closed set $F \subseteq E^c$ such that $m(F) > 0$. Again, $F \cap E$ must be non-measurable, but this is impossible since $F \cap E = \emptyset$. So we have found $m_*(E) = m_*(E^c) = 0$ and by (i), E is saturated non-measurable.

□

[1] cf. Sardella and Ziliotti 2002.

The construction (based on Armstrong 2010) relies on an interesting property of the *difference set* $D(E) = \{x - y : x, y \in E\}$ of a measurable set E with positive measure.

THEOREM A.7 (STEINHAUS, 1920). *Let $E \subseteq \mathbb{R}$ be a measurable set with $m(E) > 0$. Then the difference set $D(E)$ contains an open interval around 0.*

Proof. The proof presented here is due to Stromberg 1972. Let E be measurable with finite positive measure E. Then, for any $\varepsilon > 0$ we can find a compact set F and an open set G such that

$$m(F) + \varepsilon > m(A) > m(G) - \varepsilon.$$

In particular we can find F and G such that $2m(F) > m(G)$. Since F is compact we can find a small neighbourhood U of zero such that $F + U \subseteq G$. If for some $x \in U$ the sets $F + x$ and F were disjoint, we would have had $m(F + x) + m(F) = 2m(F) > m(G)$. This is impossible since both $F + x$ and F are subsets of G. Thus the intersection $(F + x) \cap F$ is empty for no x and any x can be written as $x = a - b$ with $a, b \in F$. This shows that $U \subseteq F - F$ and finishes the proof. □

After this preliminary work, we turn to the construction of a saturated non-measurable set. Consider the group G generated by 1 and π (any other irrational would do). Since $\mathbb{Z} \subseteq G$, the decimal parts of two different multiples of π are also in G. But those are never equal. Otherwise $n\pi = m\pi + k$ for certain intergers n, m, k and $\pi = k/n-m$ would be rational. The unit interval thus contains infinitely many points of G and by LEMMA 2.4, G must be a dense subgroup of $\mathbb{R}$.

Let V be a (non-measurable) selector for $\mathbb{R}/G$. All elements of G are of the form $g = n + m\pi$ with $n, m \in \mathbb{Z}$ so we can split G into an 'even' and an 'odd' part:

$$G_0 = \{2n + m\pi : n, m \in \mathbb{Z}\} \quad \text{and} \quad G_1 = \{2n + 1 + m\pi : n, m \in \mathbb{Z}\}.$$

Note that G_0 is a dense subgroup of G by similar arguments. When writing $G_1 = G_0 + 1$, it becomes clear that G_1 is also dense (but no subgroup). We can now make the following claim.

THEOREM A.8. *The set $S = V + G_0$ is a saturated non-measurable set.*

Proof. To prove that $m_*(S) = 0$ it suffices to show that any measurable set $F \subseteq S$ is a null set. Let F be one and assume that $x \in G_1 \cap D(F)$. We can then write

$$x = g = (v + h) - (w + j), \qquad \text{where } g \in G_1 \text{ and } h, j \in G_0,$$

to observe that $v - w = g - h + j \in G$. Since V is a selector of $\mathbb{R}/G$ we must have $v = w$. Consequently $g = h - j$, which is impossible since G_0 is closed under addition. We conclude that $D(F)$ contains no points of G_1. In particular, $D(F)$ contains no intervals because any interval would contain a point of the dense set G_1. It follows from Theorem A.7 that F is a null-set and $m_*(S) = 0$.

Now note that $V + G$ covers the whole of $\mathbb{R}$. It follows that the complement of S is precisely $(V + G) \setminus (V + G_0) = V + G_1$ and the following holds:

$$S^c = V + G_1 = V + G_0 + 1 = S + 1.$$

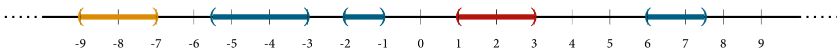
For any measurable subset A of S^c, $A - 1$ is a measurable subset of S, hence of inner measure 0. Thus $m_*(S^c) = 0$ and we conclude by Theorem A.6 that S is saturated nonmeasurable.

□

Popular Summary

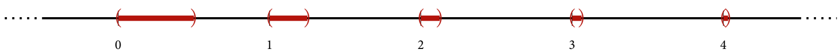
Everyone probably has an idea of what length is. As it happens to be, mathematicians did not fully grasp this notion until as late as 1902! And when they did finally grasp it, the notion turned out to be so subtle that there are things, sets, which cannot be said to have a length. That is what this text is about.

So what is length? Well, at least we do know what the length of an *interval* is. Intervals are parts of the number line. For example, $(1, 3)$ indicates the part of the number line containing all points between 1 and 3, marked in red below:



So what is the length of $(1, 3)$? Well, how about the difference between the endpoints: $3 - 1 = 2$? If you look at the yellow interval, it has the same length: $-7 - -9 = 2$. That is no surprise, because the yellow interval is the same as the red interval, after we moved it 10 steps to the right and surely, movements do not change length. We have found the first property of length: it does not change when we move things around. (Mathematicians will try to sound interesting by saying that length is *invariant under translations*.)

Now look at the blue set (a set is just any part of the the number line). Our blue set consists of three intervals: $(-5.5, -3)$, $(-2, -1)$ and $(6, 7)$ and as a result, the we would say that the size of the blue set is $2.5 + 1 + 1 = 4.5$. Of course, we can only add up these lengths, because the three intervals do not overlap. Apprently, the length of some non-overlapping intervals, taken together, should equal the sum of all their lengths. But we can also do this an infinitely many times: look at the following more difficult set, which continues forever:



Here the leftmost interval has length $\frac{1}{2}$, the second length $\frac{1}{4}$, then $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$ and so on. In this way the n'th interval will have length $\frac{1}{2^n}$. Earlier, we saw that moving things around doesn't change the length, so let's glue them all together, and zoom in (alternating colours for readability)

This picture shows that the intervals

$$\left(0, \frac{1}{2}\right), \quad \left(\frac{1}{2}, \frac{3}{4}\right), \quad \left(\frac{3}{4}, \frac{7}{8}\right), \quad \left(\frac{7}{8}, \frac{15}{16}\right), \quad \left(\frac{15}{16}, \frac{31}{32}\right), \ldots$$

together build op the unit interval $(0, 1)$. And indeed we can show what the image suggests: the sum of all their lengths equals their total length:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \cdots = 1$$

This is the third property of length: if for every natural number we take a set that does not overlap with any of the others, then all sets together (their *union*, we would say) has the same length as the sum of all the lengths of the parts. Symbolically, if for every natural number n, A_n is a set not overlapping with another A_m, then the length of the union $A_1 \cup A_2 \cup A_3 \cup \ldots$ equals the sum of the lengths:

$$\ell(A_1 \cup A_2 \cup A_3 \cup \ldots) = \ell(A_1) + \ell(A_2) + \ell(A_3) \cdots$$

Here the little cup $\cup$ signifies that we *union* two sets, that is: that we take them together.

So far we have found two properties of length

(M1) We can move sets around without changing their length, and

(M2) The length of a collection of non-overlapping parts, equals the sum of the lengths of the parts.

After identifying some useful properties, mathematicians often take these properties to be the *definition* of a new, more abstract concept. That is what we do now. If we have some operation that assigns to a set a certain positive number in such a way that conditions (M1) and (M2) hold, then we call this operation a *measure*, since it *measures* the "size" of a set.

One very fine mathematician, Henri Lebesgue, proposed the following problem in 1902: is there a measure that measures *every* set of numbers, no matter how exotic we choose them? He himself proposed a way to measure many, many sets, but he could not measure *all*. It was another mathematician, Giuseppe Vitali, who demonstrated a set so weird, that it escaped all attempts to be measured, even though our assumptions about measures and sizes were so straightforward! Such sets, which we cannot assign a size, or measure, we call *non-measurable*.

It is very hard, if not impossible, to form an image of what such a set would look like. This is due to its peculiar nature. In defining his set, Vitali made use of an assumption that was extremely controversial at the time: the Axiom of Choice. By 'axiom' we mean an axiom of set theory. That is: a *defining* property of a thing we call 'set', similar to how (M1) and (M2) were defining properties of something we called a measure.

Bertrand Russell once gave a good illustration of how the Axiom of Choice works. Suppose that you are given an infinite collection of pairs of shoes and someone asks you to give a way of choosing one shoe from of every pair of shoes. That, of course, is not difficult. Simply pick the left shoe from each pair, for example. But now the same person asks you to choose one sock out of an infinite collection of pairs of socks. What do we do? We cannot pick the left one, since the left and right sock are indistinguishable. "Just pick one of the two", would probably be our solution. This is where the axiom enters the scene. To *arbitrarily* choose an element (sock) out of an infinite collection of sets (pairs of socks), you need the Axiom of Choice: it says precisely that making these arbitrary choices, is allowed.

The problem is that there is no objective way in which mathematicians can decide whether or not this axiom is true, or, to put it differently, whether it should be accepted: that is open to dispute! If we do accept it, we can build the set of Vitali, by making infinitely many arbitrary choices. But after doing that, we can no longer describe what it exactly looks like since the choices were made arbitrarily.

In this thesis I explore these exotic sets: how can you build them, and why exactly are these arbitrary choices needed? It was hard to perceive such sets, but is also seems hard to perceive how this topic would ever be of any direct, practical use. After all, we can still use our ruler to measure lengths, right? Of course, the relevance of this theme is more subtle and more indirect. Mathematics cannot proceed unless we can prove that certain things work, and for that you sometimes have to pursue very fundamental questions. For example, probability theory and statistics, with their countless applications, could not have been developed without the theory of measures. No doubt, most mathematics will eventually find some place in the real world.